ERATO Conference on
Quantum Information Sciences
Tokyo, September 1–5, 2004

On SIC-POVMs and MUBs in Dimension 6

quant-ph/0406175

Markus Grassl

Arbeitsgruppe *Quantum Computing*, Prof. Beth
Institut für Algorithmen und Kognitive Systeme
Fakultät für Informatik, Universität Karlsruhe
Germany
http://iaks-www.ira.uka.de/QIV

Part of this work was supported by Landesstiftung Baden-Württemberg gGmbH (AZ 1.1322.01)

Identifying Quantum States

General Problem:

What is the best way to identify an arbitrary unknown quantum state ρ in a d-dimensional Hilbert space?

- ρ is a Hermitian matrix
  $\Longrightarrow d^2 - 1$ real parameters

- one von Neumann measurement provides $d - 1$ independent parameters
  $\Longrightarrow$ at least $d + 1$ different measurements

- general measurements (POVMs)
  $\Longrightarrow$ at least d^2 POVM elements

- goal: "maximal independence" of the measurement results

Mutually Unbiased Bases (MUBs)

- orthogonal bases $\mathcal{B}^j := \{|\psi_k^j\rangle : k = 1, \ldots, d\} \subset \mathbb{C}^d$
- basis states are "mutually unbiased":

$$|\langle\psi_k^j|\psi_m^l\rangle|^2 = \begin{cases} 1/d & \text{for } j \neq l, \\ \delta_{k,m} & \text{for } j = l. \end{cases}$$

- at most $d+1$ MUBs in dimension d
- constructions for $d+1$ MUBs only known for prime powers $d = p^e$
- lower bound [Klappenecker & Rötteler, quant-ph/0309120]:

$$N(m \cdot n) \geq \min\{N(m), N(n)\} \geq 3$$

Symmetric Informationally Complete POVMs

[J. M. Renes, R. Blume-Kohout, A. J. Scott, & C. M. Caves, quant-ph/0310075]

- POVM with d^2 rank-one elements $E_j = \Pi_j/d$ with $\Pi_j = |\phi_j\rangle\langle\phi_j|$
- The d^2 elements form a basis of $\mathbb{C}^{d\times d}$.
  $\Longrightarrow$ "informationally complete"
- expectation values $p_j = \mathrm{tr}(\rho E_j)$ "maximally independent":

$$\mathrm{tr}\left(\Pi_j \Pi_k\right) = |\langle\phi_j|\phi_k\rangle|^2 = \frac{1}{d+1} \qquad \text{for } j \neq k,$$

  $\Longrightarrow$ "symmetric"
- algebraic solutions for dimension $d = 2, 3, 4, 5, 8$ [e.g. Zauner 99]
- numerical solutions for $d \leq 45$ [Renes et al. 03]

Analogies to Finite Geometry

[W. K. Wootters, quant-ph/0406032]

affine planes	MUBs	SIC-POVMs
d^2 points	d^2 *Wigner operators*	d^2 states
$d(d+1)$ lines	$d(d+1)$ states	$d(d+1)$ operators "B_α"
lines are • parallel, or • intersect in one point	states are • orthogonal • unbiased	different cases for trace inner products of B_α and B_β
only known to exists for prime powers	constructions for prime powers	*conjectured:* *all dimensions*

Zauner's Conjecture

[G. Zauner, Dissertation, Universität Wien, 1999]

Conjecture:
For every dimension $d \geq 2$ there exists a SIC-POVM whose elements are the orbit of a rank-one operator E_0 under the Heisenberg group H_d.
What is more, E_0 commutes with an element T of the Jacobi group J_d.
The action of T on H_d modulo the center has order three.

support for this conjecture:

- algebraic solutions by [Zauner] for $d = 2, 3, 4, 5$ (only prime powers)
- numerical evidence by [Renes et al.] for $d \leq 45$

Weyl-Heisenberg Group

- Generators: $$H_d := \langle X, Z \rangle$$

  where $X := \sum_{j=0}^{n-1} |j+1\rangle\langle j|$ and $Z := \sum_{j=0}^{n-1} \omega_d^j |j\rangle\langle j|$

  ($\omega_d := \exp(2\pi i/d)$)

- Relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- Basis:

$$H_d \big/ \zeta(H_d) = \left\{X^a Z^b \colon a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

Jacobi Group (or Clifford Group)

- automorphism group of the Heisenberg group H_d, i.e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of J_d on H_d modulo phases corresponds to the symplectic group $SL(2, \mathbb{Z}_d)$, i.e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where} \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix}, \tilde{T} \in SL(2, \mathbb{Z}_d)$$

- J_d is generated by the discrete Fourier transform and a diagonal matrix "with quadratic phases" (depends on d odd or even)

SIC-POVM in Dimension 6

Ansatz 1:

SIC-POVM that is the orbit under H_d, i.e.,

$$|\phi_{a,b}\rangle := X^a Z^b |\phi_0\rangle \tag{1}$$

$$\langle\phi_{a,b}|\phi_{a',b'}\rangle = \begin{cases} 1 & \text{for } (a,b) = (a',b'), \\ 1/(d+1) & \text{for } (a,b) \neq (a',b') \end{cases} \tag{2}$$

$$|\phi_0\rangle = \sum_{j=0}^{d-1} (x_{2j} + i x_{2j+1}) |b_j\rangle,$$

($x_0, \ldots, x_{2d-1}$ are real variables, $x_1 = 0$)

$\Longrightarrow$ polynomial equations for $2d-1$ variables, but too complicated for $d = 6$

SIC-POVM in Dimension 6 (cntd.)

Ansatz 2:

SIC-POVM that is the orbit under H_d,

additionally: $|\phi_0\rangle$ lies in a (degenerate) ℓ-dimensional eigenspace of some $T \in J_d$

$$|\phi_0\rangle = \sum_{j=0}^{\ell-1} (x_{2j} + ix_{2j+1})|b_j\rangle,$$

here: $\ell = 3$, i.e., only 5 variables

$\Longrightarrow$ algebraic solution computed using Magma

- 144 complex solutions for the real variables
  $\Longrightarrow$ only the real solutions are valid
- in total 96 “different” such SIC-POVMs, but all these SIC-POVMs are related by complex conjugation or a global basis change

MUBs and Unitary Error Bases

[S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, & F. Vatan, quant-ph/0103162]

Theorem:

There exists k MUBs in dimension d if and only if there are $k(d-1)$ traceless, mutually orthogonal matrices $U_{j,t} \in U(d,\mathbb{C})$ that can be partitioned into k sets of commuting matrices:

$$\mathcal{B} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_k, \qquad \text{where } \mathcal{C}_j \cap \mathcal{C}_l = \emptyset \text{ and } |\mathcal{C}_j| = k-1$$

Each of the k orthogonal bases are the common eigenstates of the commuting matrices in one class $\mathcal{C}_j$.

Ansatz:

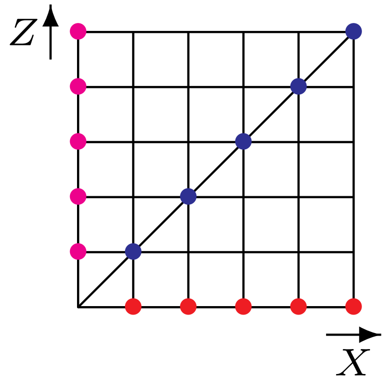
Use the matrices $X^a Z^b$ of the Weyl-Heisenberg group.

Three MUBs in any Dimension

consider the operators

$$\{X^a : a = 1, \ldots, d-1\}, \quad \{Z^a : a = 1, \ldots, d-1\}, \quad \{X^a Z^a : a = 1, \ldots, d-1\}$$

- all matrices are mutually orthogonal, the sets are disjoint, the matrices within each set commute
- geometric picture:



$\Longrightarrow$ the eigenvectors of X, Z, and XZ form three MUBs in any dimension

More than 3 MUBs in Dimension 6?

Ansatz 1:

- start with the eigenvectors of X, Z, and XZ
- search for a vector $|\psi\rangle$ that is unbiased w.r.t. these 18 vectors

$\Longrightarrow$ The system of polynomial equations has no solution.

Ansatz 2:

- start with the eigenvectors of X and Z
- search for a vector $|\psi\rangle$ that is unbiased w.r.t. these 12 vectors
- w.l.o.g, the first coordinate is $1/\sqrt{6}$

$\Longrightarrow$ There are exactly 48 solutions for $|\psi\rangle$.

The 48 Solutions

- Each solution is unbiased with respect to either 4 or 12 other vectors.
- There are 16 subsets of size 6 that are an orthonormal bases.
- No vector is unbiased with respect to one of the 16 bases.

Consequence:
Starting with the eigenvectors of X and Z, we get no more than 3 MUBs in dimension 6.

action of the Jacobi group & geometric interpretation $\Longrightarrow$:

Corollary:
Starting with the eigenvectors of “two lines that intersect only in the origin”, we get no more than 3 MUBs in dimension 6.

Why Dimension 6?

- $6 = 2 \cdot 3$ is the smallest non-prime power.
- There is no affine plane of order six.
- There are no two mutually orthogonal Latin squares (MOLS).
  $\Longrightarrow$ This *could* imply that there a no more then 3 MUBs.
- There are no more than 3 MUBs that are related to the Weyl-Heisenberg group respectively $\mathbb{Z}_6 \times \mathbb{Z}_6$.

but: A SIC-POVM in dimension 6 exists!

Conclusion/Outlook

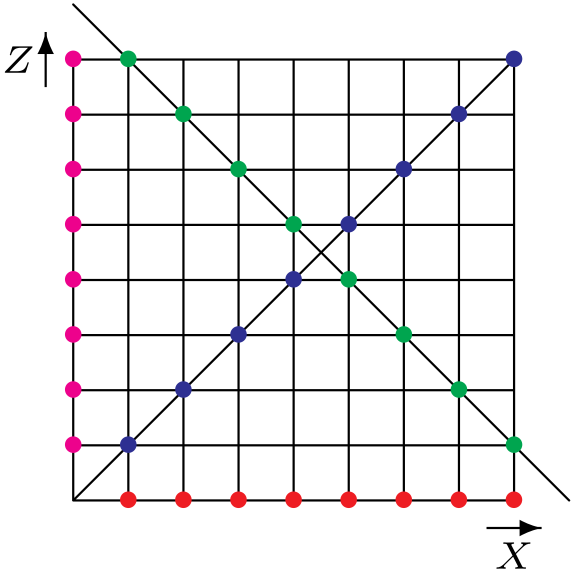
- First proof that there is a SIC-POVM in a non-prime-power dimension. (Also algebraic solutions for $d = 7, 8, 9$, but not yet for $d = 10$.)
- The connection to finite geometry is too week to exclude the existence of a SIC-POVM in dimension 6.
- results on “unextendable” MUBs

still open: Are there more than 3 MUBs in dimension 6?

other dimensions:

- $d = 4$: The eigenvectors of X, Z, and XZ form a set of three unextendable MUBs (thanks to W. Wootters for asking the question about $d = 4$).
- $d = 9$: 4 MUBs from the eigenvectors of X, Z, XZ, and XZ^{-1}

$d = 9$



$$\{X^a : a = 1, \ldots, d-1\} \qquad \{Z^a : a = 1, \ldots, d-1\}$$

$$\{X^a Z^a : a = 1, \ldots, d-1\} \qquad \{X^a Z^{-a} : a = 1, \ldots, d-1\}$$

SIC-POVM for $d = 6$

$$
\begin{aligned}
v_1 := \Bigg(\Bigg(&336(\sqrt{7}-\sqrt{21})\theta_1 - 42\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} - 378\Bigg)\theta_2^2 \\
&+\Bigg(56(3\sqrt{7}-2\sqrt{3}+3)\theta_1 + 3\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} + 63\Bigg)\theta_2 \\
&+(168 - 24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7})\theta_1 + 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} - 6\Bigg)i \\
&+\Bigg(336(\sqrt{7}+\sqrt{21})\theta_1 + 42\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} + 378\Bigg)\theta_2^2 \\
&+\Bigg(56(3\sqrt{7}-2\sqrt{3}-3)\theta_1 - 3\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} - 63\Bigg)\theta_2 \\
&+(24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7} - 168)\theta_1 - 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} + 6, \\
v_2 := \Bigg(\Bigg(&672(\sqrt{7}-\sqrt{21})\theta_1 - 168\sqrt{3} + 504\Bigg)\theta_2^2 \\
&+\Bigg(28(3\sqrt{21}+5\sqrt{3}-3\sqrt{7}-15)\theta_1 - 42\sqrt{3} + 126\Bigg)\theta_2 \\
&+(336 - 48\sqrt{21} - 112\sqrt{3} + 48\sqrt{7})\theta_1 - 12\sqrt{21} - 12\sqrt{3} + 12\sqrt{7} + 36\Bigg)i \\
&-(84\sqrt{21} - 252\sqrt{3} - 252\sqrt{7} + 252)\theta_2^2 \\
&+(84(\sqrt{21}+\sqrt{3}-3\sqrt{7}-1)\theta_1 - 6\sqrt{21} + 18\sqrt{7})\theta_2 - 24\sqrt{3} + 24, \\
v_3 := {}&6(\sqrt{7}-\sqrt{3})i + 6\sqrt{21} + 12\sqrt{3} - 12\sqrt{7} - 18
\end{aligned}
$$

SIC-POVM for $d = 6$ (cntd.)

$$
\begin{aligned}
v_4 := & \Bigg(\Bigg(336(\sqrt{7} - \sqrt{21})\theta_1 + 126\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} + 126 \Bigg) \theta_2^2 \\
& + \Bigg(56(6 - 3\sqrt{21} - 2\sqrt{3} + 3\sqrt{7})\theta_1 - 9\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} + 63 \Bigg) \theta_2 \\
& + ((168 - 24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7})\theta_1 + 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} - 54) \Bigg) i \\
& + \Bigg(336(\sqrt{21} - 3\sqrt{7})\theta_1 + 42\sqrt{21} - 378\sqrt{3} - 126\sqrt{7} + 378 \Bigg) \theta_2^2 \\
& + \Bigg(168(\sqrt{3} - 1)\theta_1 - 3\sqrt{21} + 63\sqrt{3} + 9\sqrt{7} - 63 \Bigg) \theta_2 \\
& + (24\sqrt{21} + 168\sqrt{3} - 72\sqrt{7} - 168)\theta_1 + 6 - 6\sqrt{21} - 6\sqrt{3} + 18\sqrt{7}, \\
v_5 := & \Bigg(\Bigg(672\sqrt{7}\theta_1 + 84\sqrt{21} - 168\sqrt{3} + 252 \Bigg) \theta_2^2 \\
& - \Bigg((84\sqrt{21} - 140\sqrt{3} + 84\sqrt{7} - 84)\theta_1 - 6\sqrt{21} + 42\sqrt{3} \Bigg) \theta_2 \\
& - (112\sqrt{3}\theta_1 - 48\sqrt{7}\theta_1 + 12\sqrt{3} - 12\sqrt{7} + 24) \Bigg) i \\
& + (672\sqrt{7}\theta_1 - 84\sqrt{21} - 168\sqrt{3} - 252)\theta_2^2 \\
& + \Bigg((84\sqrt{21} + 140\sqrt{3} - 84\sqrt{7} - 84)\theta_1 - 6\sqrt{21} - 42\sqrt{3} \Bigg) \theta_2 \\
& - 112\sqrt{3}\theta_1 + 48\sqrt{7}\theta_1 - 12\sqrt{3} + 12\sqrt{7} + 24, \\
v_6 := & 6(\sqrt{7} - \sqrt{3})i - 6\sqrt{21} + 18.
\end{aligned}
$$