

Lecture 3: Quantum Channels

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3.1 reduced density matrix / partial trace

Let ρ be a density matrix of a system

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{D}_1 = \{|i\rangle_1 : i \in I_1\}$ and $\mathcal{D}_2 = \{|j\rangle_2 : j \in I_2\}$ are orthonormal bases of \mathcal{H}_1 and \mathcal{H}_2 .

Then ρ can be expressed with respect to the canonical basis as

$$\rho = \sum_{\substack{i, i' \in I_1 \\ j, j' \in I_2}} s_{i,j,i',j'} |i\rangle\langle i'| \otimes |j\rangle\langle j'|$$

The reduced density matrix ρ_1 of the first system is given by

$$\rho_1 = \sum_{i, i' \in I_1} \left(\sum_{j \in I_2} s_{i,j,i',j} \right) |i\rangle\langle i'|$$

The mapping $\rho \mapsto \rho_1 =: \text{Tr}_2(\rho)$ is the partial trace of \mathcal{H} w.r.t. \mathcal{H}_2 .

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3.2 "purification", extension to a pure state

Any mixed state ρ on a system \mathcal{H}_n of dimension n can be written as the partial trace of pure state on $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim \mathcal{H}_2 = m = \dim \mathcal{H}_n$.

Consider the spectral decomposition of ρ as

$$\rho = \sum_{i=1}^m d_i |i\rangle\langle i|$$

Pick any ONB \mathcal{B}_2 of \mathcal{H}_2 and set

$$|\psi\rangle = \sum_{i=1}^m \sqrt{d_i} |i\rangle_1 |b_i\rangle_2$$

$$\Rightarrow \text{Tr}_2 (|\psi\rangle\langle\psi|) = \rho.$$

3.3 Schmidt - Decomposition

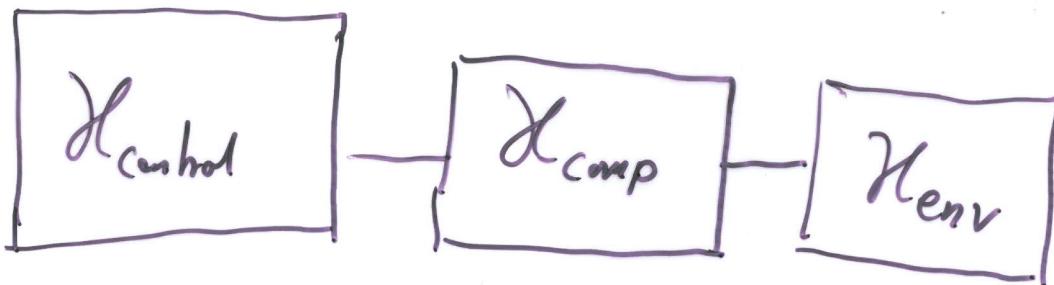
Any pure state on $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as

$$|\psi\rangle = \sum_{i=1}^m \alpha_i |b_i\rangle_1 |b'_i\rangle_2.$$

(only m instead of m^2 parameters)

3.3 (Simple) Model of a Quantum System

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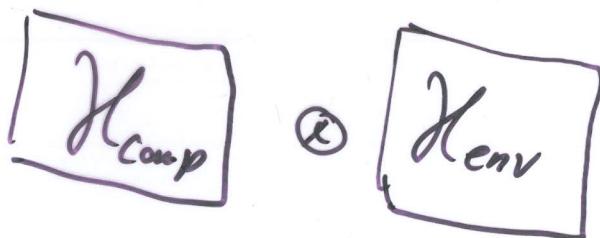
$$\mathcal{H} = \mathcal{H}_{\text{comp}} \circledast \mathcal{H}_{\text{control}} \circledast \mathcal{H}_{\text{env}}$$

perfect control \Rightarrow no entanglement between $\mathcal{H}_{\text{comp}}$ and $\mathcal{H}_{\text{control}}$

perfect shielding \Rightarrow no entanglement between $\mathcal{H}_{\text{comp}}$ and the environment \mathcal{H}_{env}

but: This cannot be achieved perfectly.

- \Rightarrow Attribute all imperfections to the environment
- \Rightarrow simplified system



Common assumptions:

- Initially, the state of system $\mathcal{H}_{\text{comp}}$ is not entangled with the environment.
- By enlarging the Hilbert space of the environment we can assume that the initial state of the environment is a pure state.

For the moment, assume that the state of the system is also pure, i.e. the initial state is

$$|\psi_0\rangle = |\psi\rangle_{\text{comp}} \otimes |\phi\rangle_{\text{env}}$$

Furthermore, we assume that the environment is sufficiently large to model the whole system as a closed system with unitary dynamics, i.e. we get a state

$$|\psi_1\rangle = U|\psi_0\rangle$$

$$\begin{aligned} \Rightarrow S_{\text{comp}} &= \text{Tr}_{\text{env}} (|\psi_1\rangle \langle \psi_1|) \\ &= \text{Tr}_{\text{env}} (U(|\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi|) U^*) \end{aligned}$$

Fix some basis $\mathcal{B}_{\text{env}} = \{|e\rangle : e \in I_{\text{env}}\}$

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$$\Rightarrow S_{\text{comp}} = \sum_{e \in I_{\text{env}}} \langle e | \phi \rangle U (|q\rangle \langle q| \otimes |\phi\rangle \langle \phi|) U^* |e\rangle$$

where $|e\rangle$ should be read $I_{\text{comp}} \otimes |e\rangle$

Define operator A_e as

$$A_e := \langle e | U | \phi \rangle$$

submatrix of the matrix U corresponding to the states $|\phi\rangle$ and $|e\rangle$, acting on \mathcal{H}_1

$$\Rightarrow S_{\text{comp}} = \sum_{e \in I_{\text{env}}} A_e |q\rangle \langle q| A_e^*$$

by linearity we get a map on mixed states on \mathcal{H}_1 :

$$S_{\text{comp}} = \sum_{e \in I_{\text{env}}} A_e S_0 A_e^*$$

3.4 General Transformations:

Completely positive maps

see: B. Schumacher, PRA 54, 2614 (1996)

properties of E : $\rho \mapsto \rho' = E(\rho)$

- i) E is linear, i.e. $E(p_1 \rho_1 + p_2 \rho_2) = p_1 E(\rho_1) + p_2 E(\rho_2)$
- ii) E is trace preserving, i.e. $\text{Tr}(E(\rho)) = \text{Tr}(\rho)$
- iii) E is positive, i.e.
if $\rho \geq 0$ then $E(\rho) \geq 0$
- iv) E is completely positive, i.e., the mapping
 $I \otimes E$
is positive for any extension of E to
a composed system.

Representations of completely positive maps

\Leftrightarrow Any C.P. map on \mathcal{H} can be represented as

a) $G(g) = \sum_{i \in I} A_i g A_i^+$

where $\sum_{i \in I} A_i^+ A_i = I$

If $\dim \mathcal{H} = d < \infty$, then we have a representation with at most d^2 terms.

b) "unitary representation"

There exists a pure state $| \phi \rangle \in \mathcal{H}'$ and a unitary transformation U on $\mathcal{H} \otimes \mathcal{H}'$ such that

$$G(g) = \text{Tr}_{\mathcal{H}'}(U(g \otimes (\langle \phi | \phi \rangle)) U^*)$$

If $\dim \mathcal{H} = d < \infty$, then $\dim \mathcal{H}' \leq d^2$.

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Remark: Changing the basis of the auxiliary system \mathcal{H}' results in a different representation of the very same map E .

Btw.: The representation is unique up to such a change of basis (see Schumacher).

The operators A_i in the operator sum repr.

$$E(g) = \sum_i A_i g A_i^*$$

are the so-called Kraus-operators.

Def.: Quantum channel

A quantum channel is a completely positive, trace preserving linear map from \mathcal{H}_m to \mathcal{H}_{out} given by

$$g \mapsto Q(g) = \sum_{i \in I_Q} A_i g A_i^*$$

The operators A_i are called the error-operators of the channel Q .

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Remark: Previously, a F.p.-map was acting on \mathcal{H} , i.e. the image was the same as the input space.

For quantum channels, we allow $\mathcal{H}_{in} \neq \mathcal{H}_{out}$.

For this, one can consider $\mathcal{H}_{in} \otimes \mathcal{H}_{out}$

$$Q(S_m) = \text{Tr}_{in} \left(\tilde{E} \left(S_{in} \otimes |0\rangle\langle 0| \right) \right) = S_{out}$$

Product channel

Given quantum channels Q_1 and Q_2 with error operators $\{A_i : i \in I_{Q_1}\}$ and $\{B_j : j \in I_{Q_2}\}$ then we can define the product channel $Q_1 \otimes Q_2$ as

$$\begin{aligned} g \mapsto (Q_1 \otimes Q_2)(g) &= g'_{12} \\ &= \sum_{i \in I_{Q_1}} \sum_{j \in I_{Q_2}} (A_i \otimes B_j) g (A_i \otimes B_j)^+ \end{aligned}$$

so the error operators are given by $A_i \otimes B_j$.

For $Q_1 = Q_2$, we get $Q^{\otimes 2} := Q \otimes Q$,
or more general $Q^{\otimes n}$.

"Unitary representation" of $Q_1 \otimes Q_2$:

$$g'_{12} = \text{Tr}_{\text{env}_1, \text{env}_2} \left(U_{Q_1}^{(1, \text{env}_1)} \otimes U_{Q_2}^{(2, \text{env}_2)} (g^{(1,2)} \otimes |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|) U^+ \right)$$

If the initial state of the combined environment $\mathcal{H}_{\text{env}_1} \otimes \mathcal{H}_{\text{env}_2}$ is not a product state,

we can express it as

$$|\phi\rangle_{\text{env}_1, \text{env}_2} = U^{(\text{env}_1, \text{env}_2)} (|\phi_1\rangle \otimes |\phi_2\rangle)$$

\Rightarrow the unitary operation describing the channel is

now

$$U_{Q_1}^{(1, \text{env}_1)} \cdot U_{Q_2}^{(2, \text{env}_2)} \cdot U^{(\text{env}_1, \text{env}_2)}$$

In general, this is not a tensor product w.r.t.
to $(\mathcal{X}_1 \otimes \mathcal{H}_{\text{env}}) \otimes (\mathcal{X}_2 \otimes \mathcal{H}_{\text{env}})$

\Rightarrow The operators $A^{(x_2)}$ obtained by taking
the partial trace wrt $\mathcal{H}_{\text{env}_1} \otimes \mathcal{H}_{\text{env}_2}$
are no longer tensor products.

If the initial states of the environments are correlated/entangled, the channel uses are no longer independent (\Rightarrow same kind of channel "memory" despite the fact that we are using a different environment for each transmission).