Lecture 3: Quantum Channels

3.1 reduced density matrix / partial trace

Let ρ be a density matrix of a system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{B}_1 = \{ |i\rangle_1 : i \in I_1 \}$ and $\mathcal{B}_2 = \{ |j\rangle_2 : j \in I_2 \}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Then ρ can be expressed with respect to the canonical basis as

$$\rho = \sum_{\substack{i,i' \in I_1 \\ j,j' \in I_2}} \rho_{ij,i'j'} \, |i\rangle\langle i'| \otimes |j\rangle\langle j'|$$

The reduced density matrix ρ_1 of the first system is given by

$$\rho_1 = \sum_{i,i' \in I_1} \left(\sum_{j \in I_2} \rho_{ij,i'j} \right) |i\rangle\langle i'|$$

The mapping $\rho \mapsto \rho_1 =: \mathrm{Tr}_2(\rho)$ is the partial trace of $\mathcal{H}$ w.r.t. $\mathcal{H}_2$.

3.2 „purification", extension to a pure state

Any mixed state ρ on a system $\mathcal{H}_1$ of dimension m can be written as the partial trace of pure state on $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim \mathcal{H}_2 = m = \dim \mathcal{H}_1$.

Consider the spectral decomposition of ρ as

$$\rho = \sum_{i=1}^{m} \lambda_i |i\rangle\langle i|$$

Pick any ONB $\mathcal{B}_2$ of $\mathcal{H}_2$ and set

$$|\psi\rangle = \sum_{i=1}^{m} \sqrt{\lambda_i}\, |i\rangle_1 |b_i\rangle_2$$

$$\Rightarrow \quad \mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \rho.$$

3.3 Schmidt-Decomposition

Any pure state on $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as $|\psi\rangle = \sum_{i=1}^{m} \alpha_i |b_i\rangle_1 |b_i'\rangle_2$.

(only m instead of m^2 parameters)

3.3 (Simple) Model of a Quantum System

$$\boxed{\mathcal{H}_{control}} - \boxed{\mathcal{H}_{comp}} - \boxed{\mathcal{H}_{env}}$$

$$\mathcal{H} = \mathcal{H}_{comp} \otimes \mathcal{H}_{control} \otimes \mathcal{H}_{env}$$

perfect control $\Rightarrow$ no entanglement between $\mathcal{H}_{comp}$ and $\mathcal{H}_{control}$

perfect shielding $\Rightarrow$ no entanglement between $\mathcal{H}_{comp}$ and the environment $\mathcal{H}_{env}$

but: This cannot be achieved perfectly.

$\Rightarrow$ Attribute all imperfections to the environment

$\Rightarrow$ simplified system

$$\boxed{\mathcal{H}_{comp}} \otimes \boxed{\mathcal{H}_{env}}$$

Common assumptions:

- Initially, the state of system $\mathcal{H}_{comp}$ is not entangled with the environment.
- By enlarging the Hilbert space of the environment we can assume that the initial state of the environment is a pure state.

For the moment, assume that the state of the system is also pure, i.e. the initial state is

$$|\psi_0\rangle = |\psi\rangle_{comp} \otimes |\phi\rangle_{env}$$

Furthermore, we assume that the environment is sufficiently large to model the whole system as a closed system with unitary dynamics, i.e. we get a state

$$|\psi_1\rangle = U|\psi_0\rangle$$

$$\Rightarrow \quad \rho_{comp} = \mathrm{Tr}_{env}\left(|\psi_1\rangle\langle\psi_1|\right)$$

$$= \mathrm{Tr}_{env}\left(U\left(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|\right) U^\dagger \right)$$

Fix some basis $\mathcal{B}_{env} = \{ |e\rangle : e \in I_{env} \}$

$$\Rightarrow \quad \rho_{comp} = \sum_{e \in I_{env}} \langle e | \Big(U \big(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| \big) U^\dagger \Big) | e \rangle$$

where $|e\rangle$ should be read $I_{comp} \otimes |e\rangle$

Define operator A_e as

$$A_e := \langle e | U | \phi \rangle$$

submatrix of the matrix U corresponding to the states $|\phi\rangle$ and $|e\rangle$, acting on $\mathcal{H}_1$

$$\Rightarrow \quad \rho_{comp} = \sum_{e \in I_{env}} A_e |\psi\rangle\langle\psi| A_e^+$$

by linearity we get a map on mixed states on $\mathcal{H}_1$:

$$\boxed{\rho_{comp} = \sum_{e \in I_{env}} A_e \, \rho_0 \, A_e^+}$$

3.4 General Transformations: Completely positive maps

see: B. Schumacher, PRA 54, 2614 (1996)

properties of E: $\rho \mapsto \rho' = E(\rho)$

i) E is linear, i.e. $E(p_1 \rho_1 + p_2 \rho_2)$
$$= p_1 E(\rho_1) + p_2 E(\rho_2)$$

ii) E is trace preserving, i.e $\mathrm{Tr}(E(\rho)) = \mathrm{Tr}(\rho)$

iii) E is positive, i.e

if $\rho \geq 0$ then $E(\rho) \geq 0$

iv) E is completely positive, i.e, the mapping
$$I \otimes E$$
is positive for any extension of E to a composed system.

Representations of completely positive maps

Any c.p. map on $\mathcal{H}$ can be represented as

a) $$\mathcal{E}(\rho) = \sum_{i \in I} A_i \rho A_i^+$$

where $$\sum_{i \in I} A_i^+ A_i = I$$

If $\dim \mathcal{H} = d < \infty$, then we have a representation with at most d^2 terms.

b) „unitary representation"

There exists a pure state $|\phi\rangle \in \mathcal{H}'$ and a unitary transformation U on $\mathcal{H} \otimes \mathcal{H}'$ such that

$$\mathcal{E}(\rho) = \mathrm{Tr}_{\mathcal{H}'}\left(U(\rho \otimes |\phi\rangle\langle\phi|) U^+ \right)$$

If $\dim \mathcal{H} = d < \infty$, then $\dim \mathcal{H}' \leq d^2$.

Remark: Changing the basis of the auxilliary system $\mathcal{H}'$ results in a different representation of the very same map $\mathcal{E}$.

But: The representation is unique up to such a change of basis (see Schumacher).

The operators A_i in the "operator sum repr."

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^+$$

are the so-called <u>Kraus-operators</u>.

<u>Def.:</u> Quantum channel

A quantum channel is a completely positive, trace preserving linear map from $\mathcal{H}_{in}$ to $\mathcal{H}_{out}$ given by

$$\rho \mapsto Q(\rho) = \sum_{i \in I_Q} A_i \rho A_i^+$$

The operators A_i are called the error-operators of the channel Q.

Remark: Previously, a c.p. map was acting on $\mathcal{H}$, i.e the image was the same as the input space.

For quantum channels, we allow $\mathcal{H}_{in} \neq \mathcal{H}_{out}$.

For this, one can consider $\mathcal{H}_{in} \otimes \mathcal{H}_{out}$

$$\mathcal{Q}(\rho_{in}) = \mathrm{Tr}_{in}\left(\tilde{E}\left(\rho_{in} \otimes |0\rangle\langle 0|\right)\right) = \rho_{out}$$

Product channel

Given quantum channels Q_1 and Q_2 with error operators $\{A_i : i \in I_{Q_1}\}$ and $\{B_j : j \in I_{Q_2}\}$ then we can define the product channel $Q_1 \otimes Q_2$ as

$$\rho \mapsto (Q_1 \otimes Q_2)(\rho) = \rho'_{12}$$

$$= \sum_{i \in I_{Q_1}} \sum_{j \in I_{Q_2}} (A_i \otimes B_j)\, \rho\, (A_i \otimes B_j)^\dagger$$

so the error operators are given by $A_i \otimes B_j$.

For $Q_1 = Q_2$, we get $Q^{\otimes 2} := Q \otimes Q$, or more general $Q^{\otimes n}$.

„unitary representation" of $Q_1 \otimes Q_2$:

$$\rho'_{12} = \mathrm{Tr}_{env_1, env_2}\left(U_{Q_1}^{(1,env_1)} \otimes U_{Q_2}^{(2,env_2)} \left(\rho^{(1,2)} \otimes |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|\right) U^\dagger \right)$$

If the initial state of the combined environment $\mathcal{H}_{env_1} \otimes \mathcal{H}_{env_2}$ is not a product state, we can express it as

$$|\phi\rangle_{env_1, env_2} = U^{(env_1, env_2)}\left(|\phi_1\rangle \otimes |\phi_2\rangle\right)$$

$\Longrightarrow$ the unitary operation describing the channel is now

$$U_{Q_1}^{(1, env_1)} \cdot U_{Q_2}^{(2, env_2)} \cdot U^{(env_1, env_2)}$$

In general, this is not a tensor product w.r.t. to

$$\left(\mathcal{H}_1 \otimes \mathcal{H}_{env_1}\right) \otimes \left(\mathcal{H}_2 \otimes \mathcal{H}_{env_2}\right)$$

$\Longrightarrow$ The operators $A^{(12)}$ obtained by taken the partial trace w.r.t $\mathcal{H}_{env_1} \otimes \mathcal{H}_{env_2}$ are no longer tensor products.

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If the initial states of the environments are correlated/entangled, the channel uses are no longer independent (⇒ some kind of channel „memory" despite the fact that we are using a different environment for each transmission).