Lecture 5: Quantum Channel Capacity
Some Quantum Channels

q-channel capacity:

$$\mathcal{Q}(Q) := \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \left\{ \frac{k}{n} : \exists K, C, D : F_p(D\, Q^{\otimes n} C) > 1-\varepsilon \right\}$$

variations:

- quantum channel capacity with two-way classical communication $\mathcal{Q}_2(Q)$
  - classical communication does not allow to transmit quantum states
- classical capacity of a quantum channel

$$\mathcal{K}(Q) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \left\{ \frac{k}{n} : \exists K, C, D\ \forall |\psi\rangle \in \{|0\rangle, |1\rangle\}^{\otimes k} \quad F_p(|\psi\rangle, D\, Q^{\otimes n} C) > 1-\varepsilon \right\}$$

$$\mathcal{Q}(Q) \le \mathcal{K}(Q)$$

- entanglement assisted capacities: sender and receiver share some maximally entangled states examples:

- teleportation.

  send a quantum state using only classical information <u>and</u> shared entanglement

  resources: $1\ \text{ebit} + 2\ \text{bit} \mathrel{\hat=} 1\ \text{qbit}$

  ↑ 1 EPR-pair ↑ 2 classical bits ↑ 1 qubit

- dense coding:

  - send one half of an EPR pair to Bob
  - Alice performs one out four operations on her qubit

    $$U = \sigma_x^{b_0}\sigma_z^{b_1} \qquad b_0, b_1 \in \{0,1\}$$

  - Alice sends her qubit to Bob
  - Bob performs a measurement in the Bell basis $\Rightarrow$ four different outcomes

  $1\ \text{ebit} + 1\ \text{qubit} \mathrel{\hat=} 2\ \text{bit}$

Some particular quantum channels

5.1 Depolarizing channel

The quantum state ρ is transmitted faithfully with probability $1-p$ or replaced by a random state with probability p.

$$\rho \mapsto D_p^{pol}(\rho) = (1-p)\cdot\rho + p\cdot\frac{\mathbb{1}}{tr(\mathbb{1})}$$

warning: The completely mixed state $\mathbb{1}/tr(\mathbb{1})$ does have a non-zero overlap with any state.

error operators for $\mathcal{H} = \mathbb{C}^2$:

$$D_p^{pol}(\rho) = \left(1-\tfrac{3}{4}p\right)\cdot\rho + \frac{p}{4}\left(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z\right)$$

$\Rightarrow$ error operators

$$\left\{\mathbb{1}\sqrt{1-3/4p},\ \sigma_x\frac{\sqrt{p}}{2},\ \sigma_y\frac{\sqrt{p}}{2},\ \sigma_z\frac{\sqrt{p}}{2}\right\}$$

5.2 Generalized Qubit-Depolarization Channel

error operators $\{ \mathbb{1}, \sigma_x, \sigma_y, \sigma_z \}$ with different weights, i.e

$$\mathcal{D}_{(p_x,p_y,p_z)}(\rho) = (1-p_x-p_y-p_z)\cdot\rho$$
$$+\, p_x\,\sigma_x\rho\sigma_x + p_y\cdot\sigma_y\rho\sigma_y + p_z\sigma_z\rho\sigma_z$$

5.3 Dephasing Channel

Szenario: Assume that the channel measures the state with respect to an ONB $\mathcal{B} = \{ |b_i\rangle : i \in I \}$

$\Rightarrow$ The pure input state $|\psi\rangle = \sum_{i\in I} \alpha_i\, |b_i\rangle$ is transformed into the mixture

$$\rho = \sum_{i\in I} |\alpha_i|^2\, |b_i\rangle\langle b_i|$$

channel:

$$\rho \to \mathcal{D}^{ph}_{p,\mathcal{B}}(\rho) := (1-p)\cdot\rho + p\sum_{i\in I} |b_i\rangle\langle b_i|\rho|b_i\rangle\langle b_i|$$

note: $|0\rangle\langle 0| = \frac{1}{2}(\mathbb{1}+\sigma_z)$, $|1\rangle\langle 1| = \frac{1}{2}(\mathbb{1}-\sigma_z)$

5.4 Results on the channel capacity

Even for the depolarizing which has a lot of symmetry and only one parameter, the channel capacity is not known (in general).

results (from Bennett, DiVincenco, Shor, Smolin etc) in 1997/98

1. For $p < 0,25408$ all of Q, Q_2, and $\mathcal{K}$ are positive.
2. $1/3 < p < 2/3$, the quantum capacity $Q = 0$, but both Q_2 and $\mathcal{K}$ are positive.
3. $2/3 \leq p < 1$, both Q and Q_2 are zero, but $\mathcal{K}$ is positive
4. For $p = 1$, $Q = Q_2 = \mathcal{K} = 0$.

The classical capacity of the depolarizing channel

(Ch. King, IEEE Trans. on Information Theory, 2003)

Szenario: Use quantum states ρ_i to send classical information over the quantum channel

$$D_p^{pol}(\rho) = (1-p)\cdot\rho + p\cdot \mathbb{1}/\mathrm{tr}(\mathbb{1})$$

Input alphabet X with some distribution
$\Rightarrow$ ensemble $\mathcal{E}$ of quantum states

Output alphabet Y corresponds to some measurements $\mathcal{M}$ (or POVM) on the output.

single use capacity:

$$\mathcal{K}(Q)^{\text{single use}} = \sup_{\mathcal{E},\mathcal{M}} I(X;Y)$$

capacity:

$$\mathcal{K}(Q) = \lim_{n\to\infty} \sup \frac{1}{n} \sup_{\mathcal{E},\mathcal{M} \text{ in } \mathcal{H}^{\otimes n}} I(X;Y)$$

for some channels: $\displaystyle \frac{1}{2} \sup_{\mathcal{E},\mathcal{M}\in\mathcal{H}^{\otimes 2}} (X^{(2)};Y^{(2)}) > \sup_{\mathcal{E},\mathcal{M}\in\mathcal{H}} (X^{(1)},Y^{(1)})$

super additivity

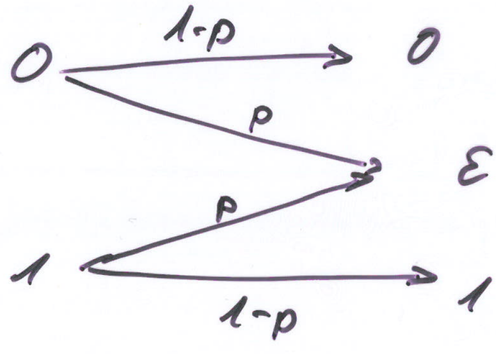
but: For the depolarizing channel, the „classical capacity" is additive, i.e. we don't need to take the limit $n \to \infty$.

Moreover, the capacity is achieved using some orthogonal pure states as inputs and measure in the same basis.

additivity: for $n > 1$, use ~~sta~~ product states and individual (product) measurements

5.5 Quantum erasure channel

recall: BEC is given by



Input: dim $\mathcal{H}_{in} = d$

Output: dim $\mathcal{H}_{out} = d+1$ using an additional orthogonal state $|\varepsilon\rangle$

$$QEC_p(\rho) = (1-p)\cdot\rho + p\cdot|\varepsilon\rangle\langle\varepsilon|$$

capacities of the QEC:

$$Q(QEC_p) = \max\{0, 1-2p\}$$

$$\mathcal{K}(QEC_p) = Q_2(QEC_p) = 1-p$$

proof: see Bennett, DiVincenzo & Smolin 97

variations:

(qubit) phase erasure channel:

input: one qubit
output: two qubits, use one qubit to indicate the error event

$$\rho \mapsto (1-p)\,\rho \otimes |0\rangle\langle 0| + p\left(\tfrac{1}{2}\rho + \tfrac{1}{2}\sigma_z \rho \sigma_z\right) \otimes |1\rangle\langle 1|$$

combination of „full erasure" and „phase erasure"

$$\rho \mapsto (1-\delta-\varepsilon)\,\rho \otimes |0\rangle\langle 0| + \varepsilon\, |e\rangle\langle e| \otimes |0\rangle\langle 0| + \delta\left(\tfrac{1}{2}\rho + \tfrac{1}{2}\sigma_z \rho \sigma_z\right) \otimes |1\rangle\langle 1|$$

capacities:

$$Q = \max\{0, 1-\delta-2\varepsilon\}$$

$$Q_2 = 1-\delta-\varepsilon$$

$$K = 1-\varepsilon$$

other types of channels

- channels arising from open quantum, including non-unitary dynamics e.g. by monitoring a system for quantum jumps
- collective effects, e.g. apply the same (local) transformation to all subsystems

$$\rho \mapsto (1-p_x-p_y-p_z)\rho + \sum_{i=x,y,z} p_i \, \sigma_i^{\otimes n} \rho \, \sigma_i^{\otimes n}$$

interchanging particles with same probability (SWAP)

$$\mathrm{SWAP} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} = \frac{1}{2}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\right)$$