Lecture 18. (Still) Encoding Circuits

18.1 Stabilizer States / Graph States

stabilizer S with $|S| = 2^n$

$\Rightarrow$ unique $+1$-eigenstate, code $C = [[n, 0, d]]$

stabilizer matrix $S = [\, X \mid Z \,]$ with $X, Z \in \mathbb{F}_2^{n \times n}$

$\mathrm{rk}(S) = n$

normal form for computing an encoding circuit. local operations H, $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$\rightarrow$ matrix X in triangular form

here: even $\underline{X = I}$ by row operations

commutation of the stabilizer $X \cdot Z^t - Z \cdot X^t = 0$

$\overset{X=I}{\Longrightarrow}$ $Z^t = Z$ (symmetric matrix)

local operations: diagonal of Z is zero

$\Rightarrow$ Z defines a undirected graph on n vertices

Example: binary code $C = [8,4,4] = C^\perp$

$$G = \begin{bmatrix} 1&0&0&0&0&1&1&1\\ 0&1&0&0&1&0&1&1\\ 0&0&1&0&1&1&0&1\\ 0&0&0&1&1&1&1&0 \end{bmatrix} \qquad G\cdot G^t = 0$$

CSS construction with $C_1 = C_2 = C = C^\perp$

one state: $\frac{1}{4}\sum_{c\in C} |c\rangle \qquad (*)$

Stabiliser $S_0 = \left[\begin{array}{c|c} G & 0\\ \hline 0 & G \end{array}\right] \in \mathbb{F}_2^{8\times 16}$

$$S_0 = \left[\begin{array}{cccccccc|cccccccc}
1&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0
\end{array}\right] \rightarrow \left[\begin{array}{cccccccc|cccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&0\\ \hline
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&0&1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0&1&0&0&0&0
\end{array}\right]$$

every column (pair) contains at least one non-zero entry

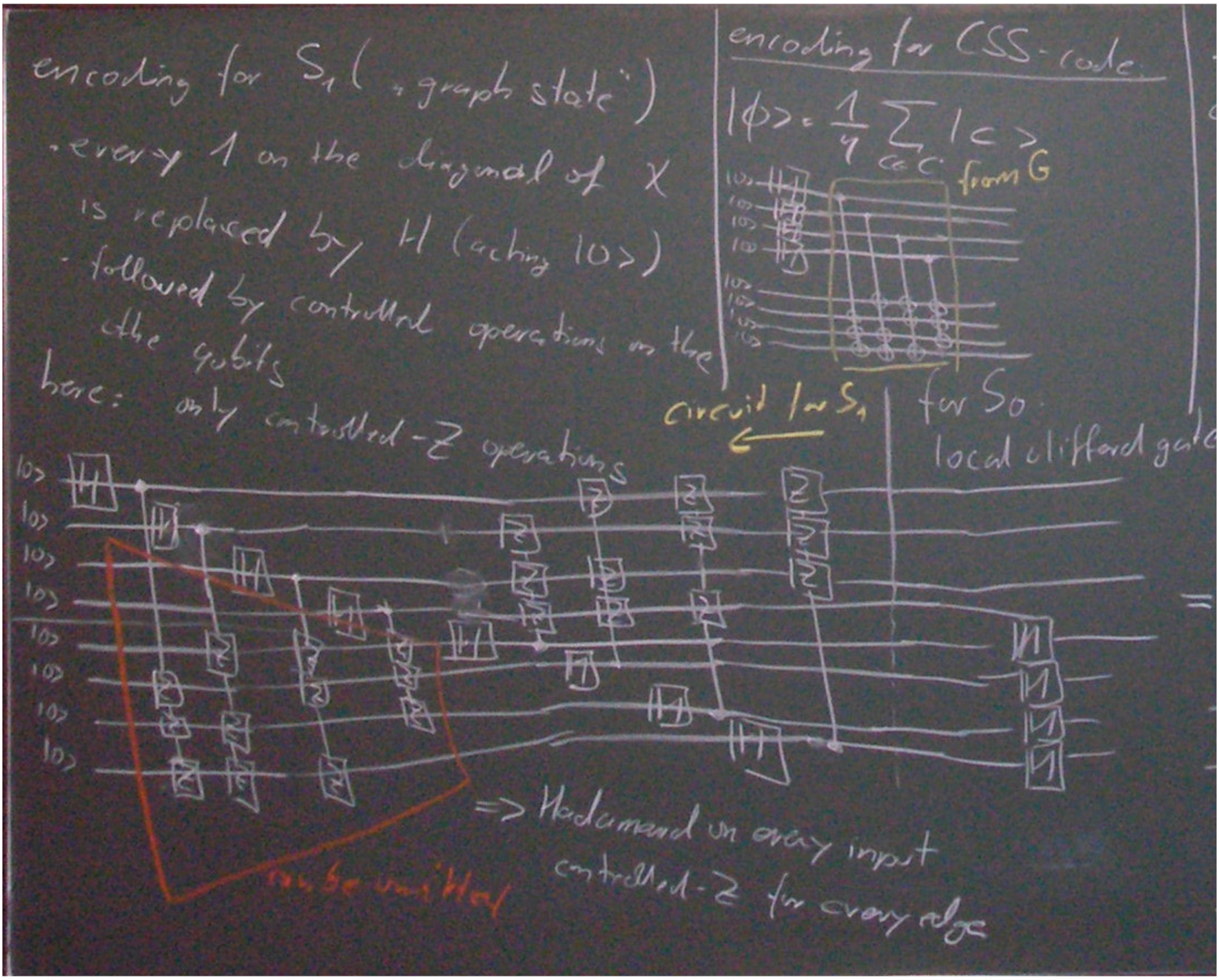
$\hookrightarrow$ local Clifford $\to \sigma_x$

Hadamard on the last 4 positions

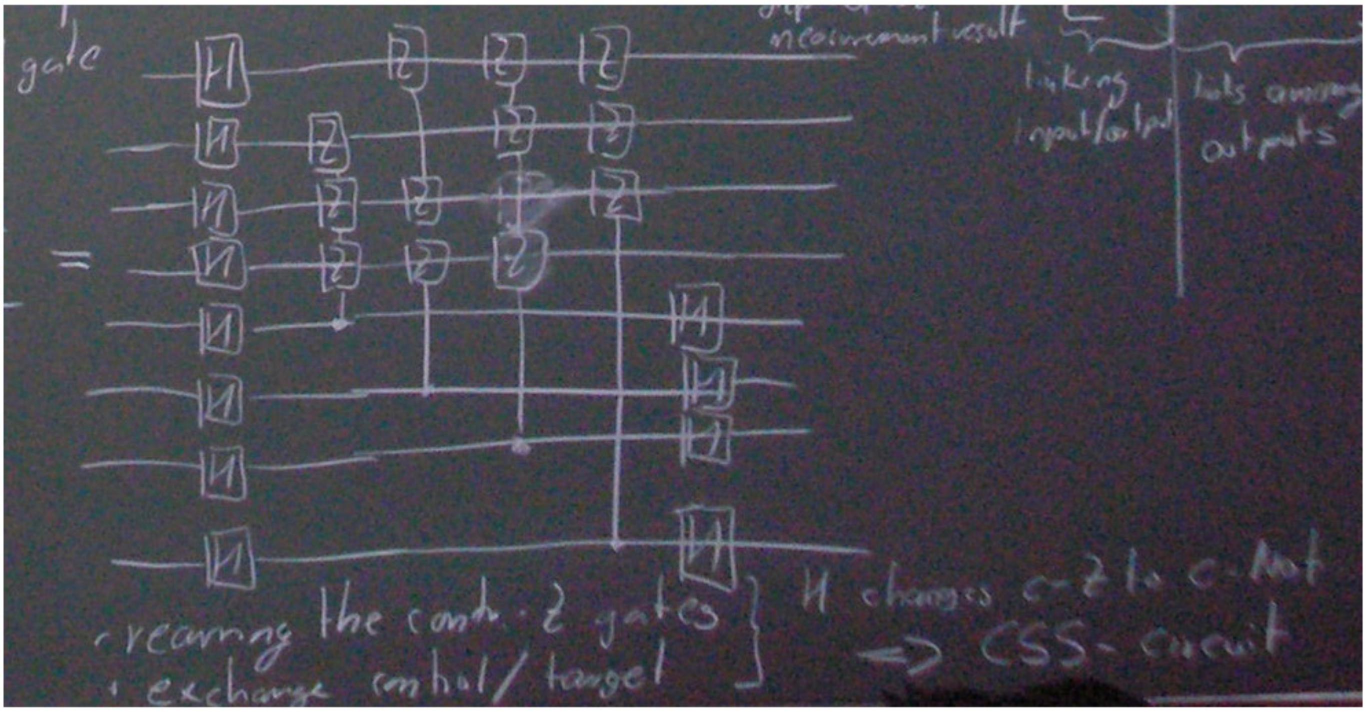
$\downarrow$ row operations

$$S_1 = \left[\begin{array}{cccccccc|cccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&0\\ \hline
0&0&0&0&1&0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&1&0&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0
\end{array}\right]$$

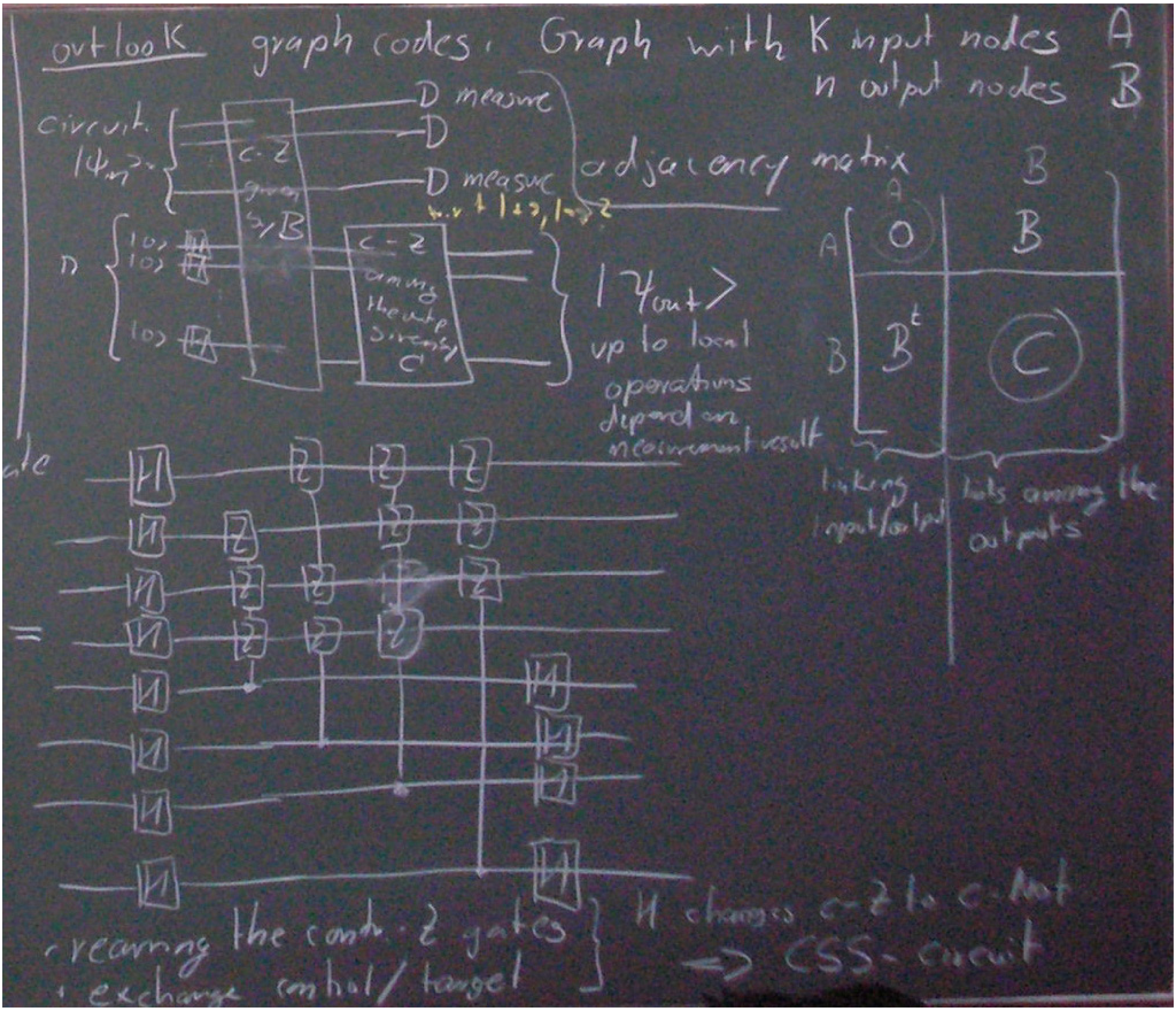
CSS-code $\Rightarrow$ graph is bi-partite

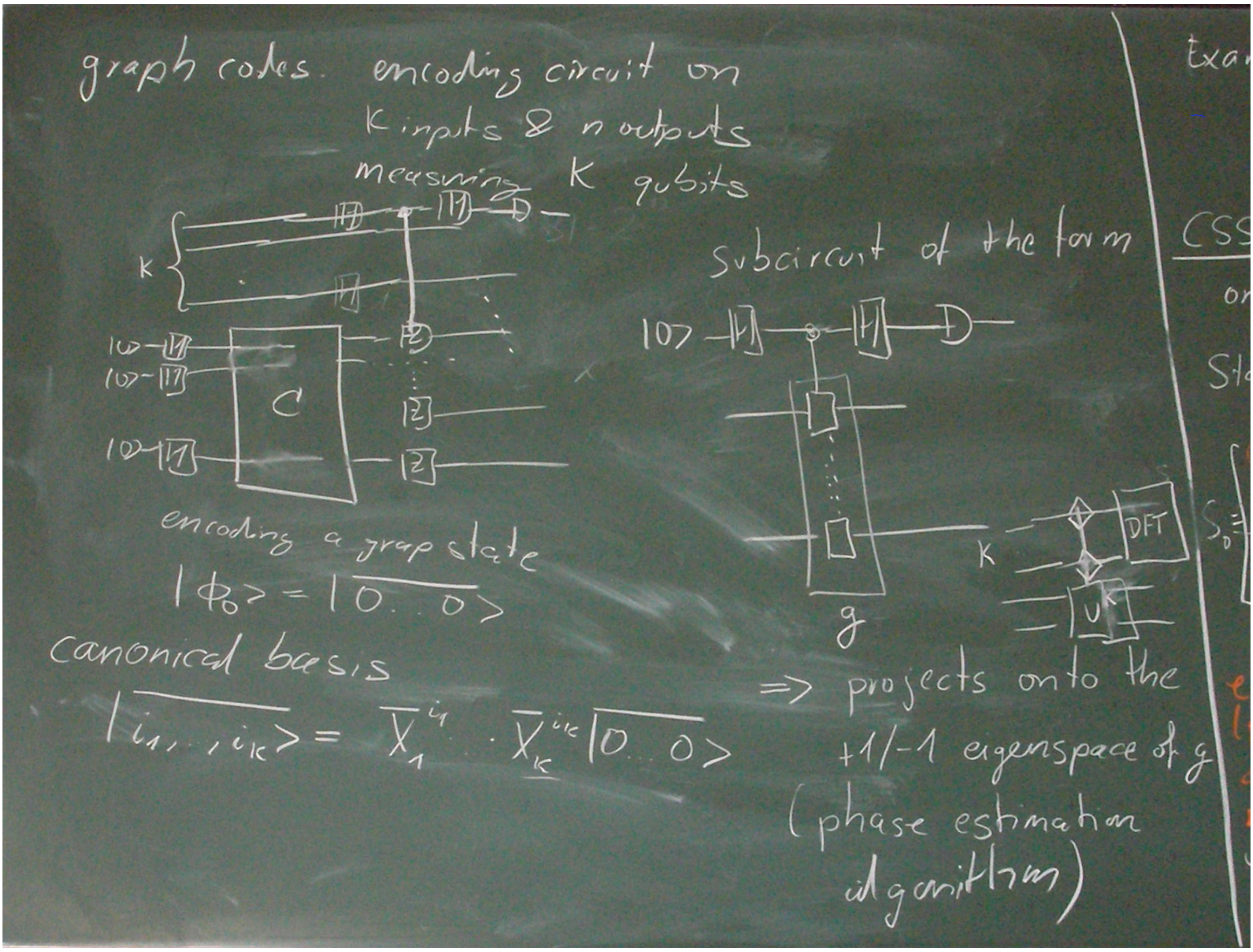


encoding for S_1 ("graph state")
· every 1 on the diagonal of X
is replaced by H (acting $|0\rangle$)
· followed by controlled operations on the other qubits
here: only controlled-Z operations
encoding for CSS-code
$|\phi\rangle = \frac{1}{4}\sum_{c\in C} |c\rangle$
from G
circuit for S_1
for S_0: local clifford gate
=> Hadamard on every input
controlled-Z for every edge
can be omitted



gate
measurement result
linking input/output
links among outputs
rearrange the contr.-Z gates
+ exchange control/target
H changes c-Z to c-Not
=> CSS-circuit


outlook
graph codes: Graph with K input nodes A
n output nodes B
circuit
D measure
D
D measure
c-Z
|0⟩
|0⟩
|0⟩
c-Z
|ψout⟩
up to local
operations
depend on
measurement result
adjacency matrix
A
B
0
B
B^t
C
linking input/output
links among the outputs
H
Z
rearrange the contr.-Z gates
+ exchange control/target
H changes c-Z to c-Not
⟺ CSS-circuit


graph codes. encoding circuit on
k inputs & n outputs
measuring k qubits
k
|0> H
C
Z
encoding a grap state
|Φ0> = |0...0>
canonical basis
|i1,...,ik> = X1^i1 ... Xk^ik |0...0>
subcircuit of the form
|0> H H
g
k
DFT
U^k
=> projects onto the
+1/-1 eigenspace of g
(phase estimation
algorithm)

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