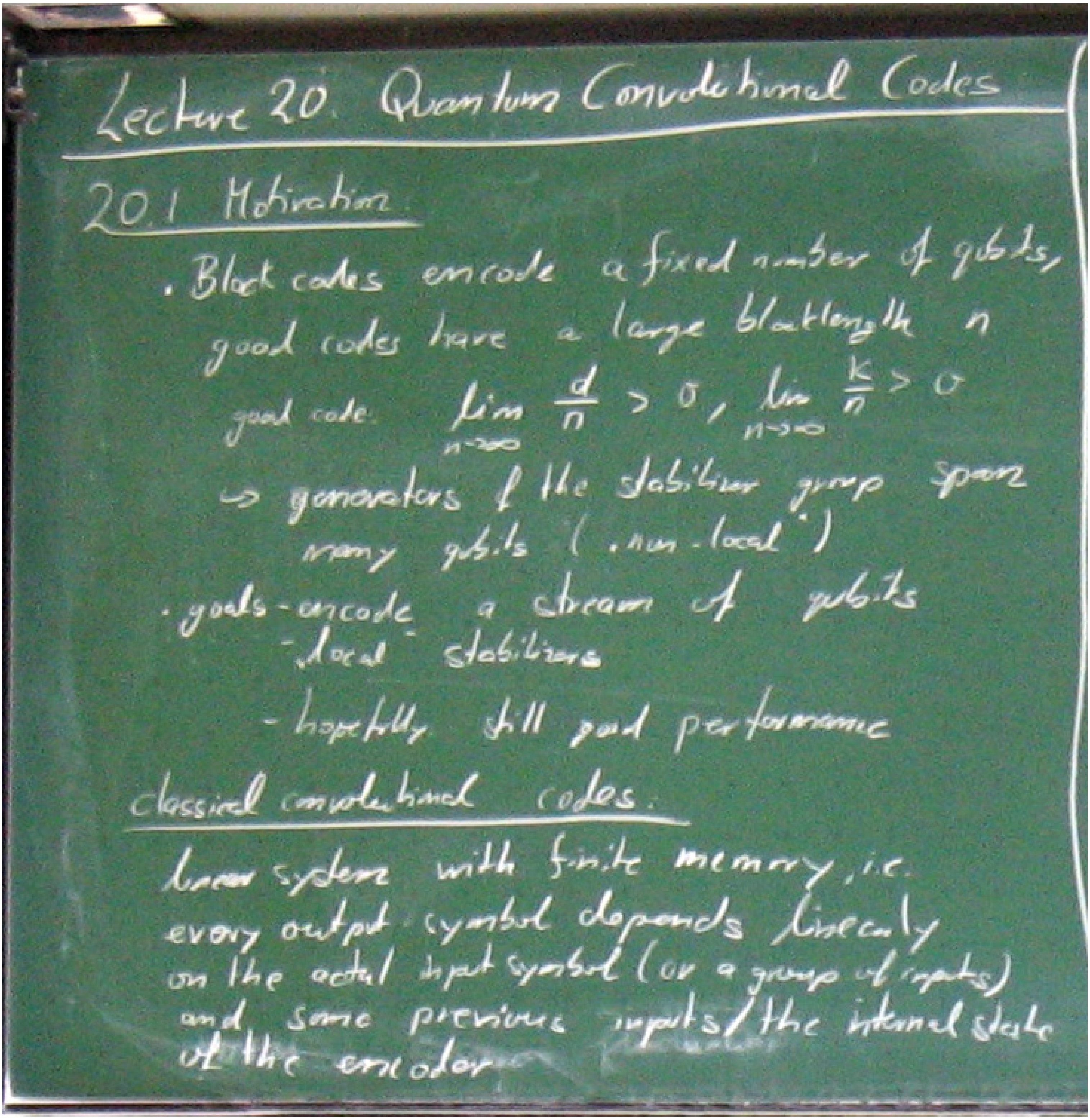
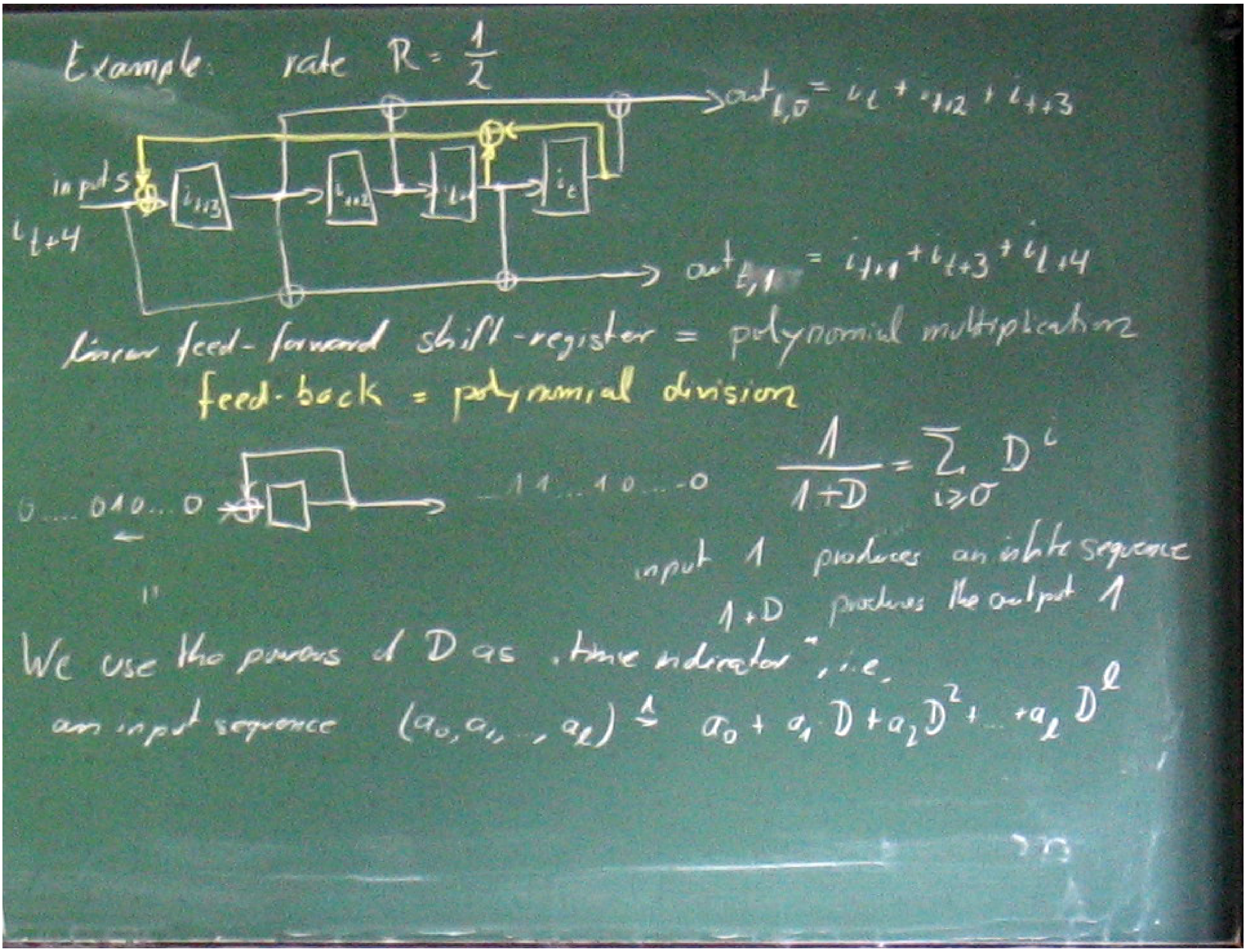
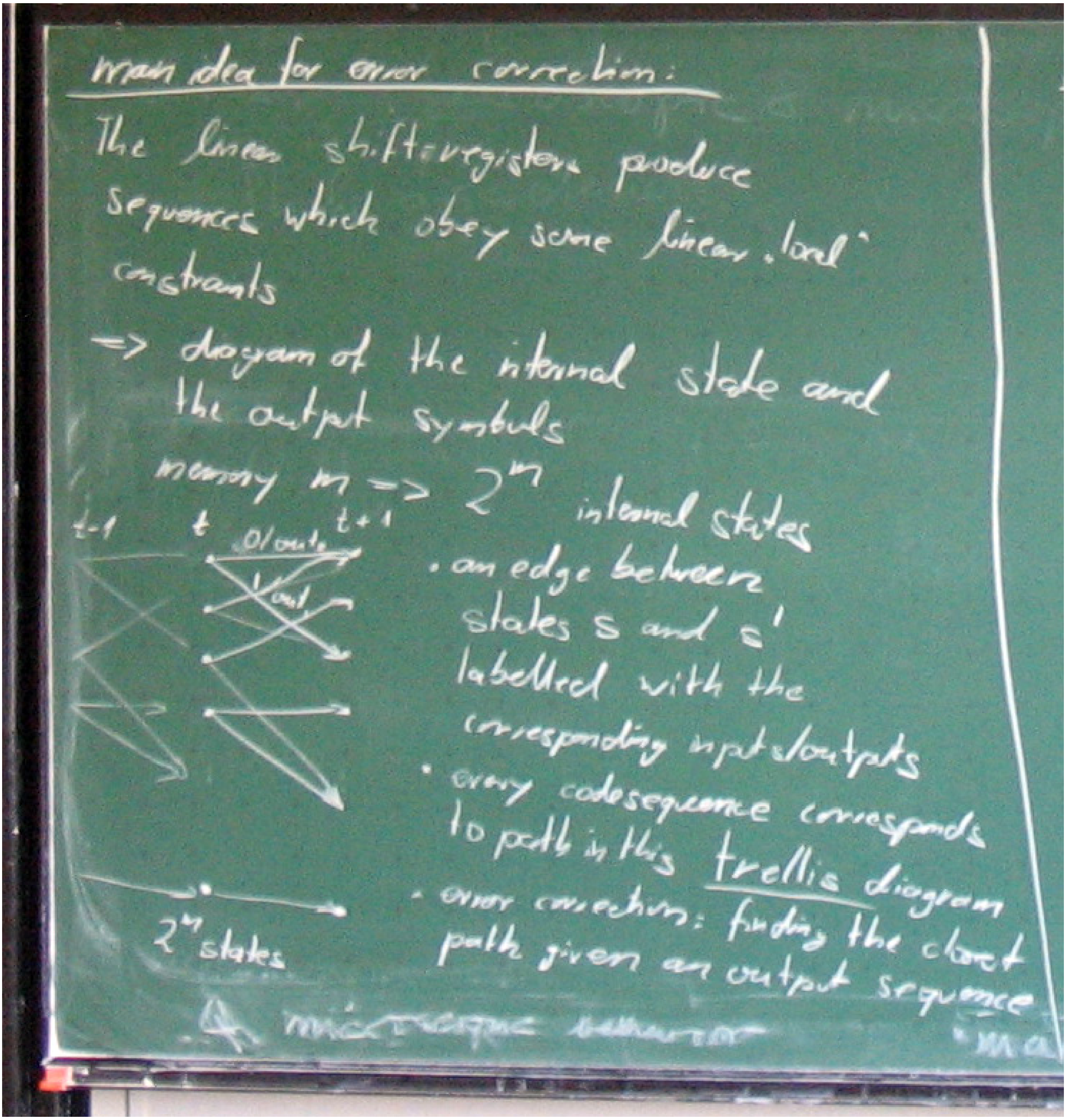
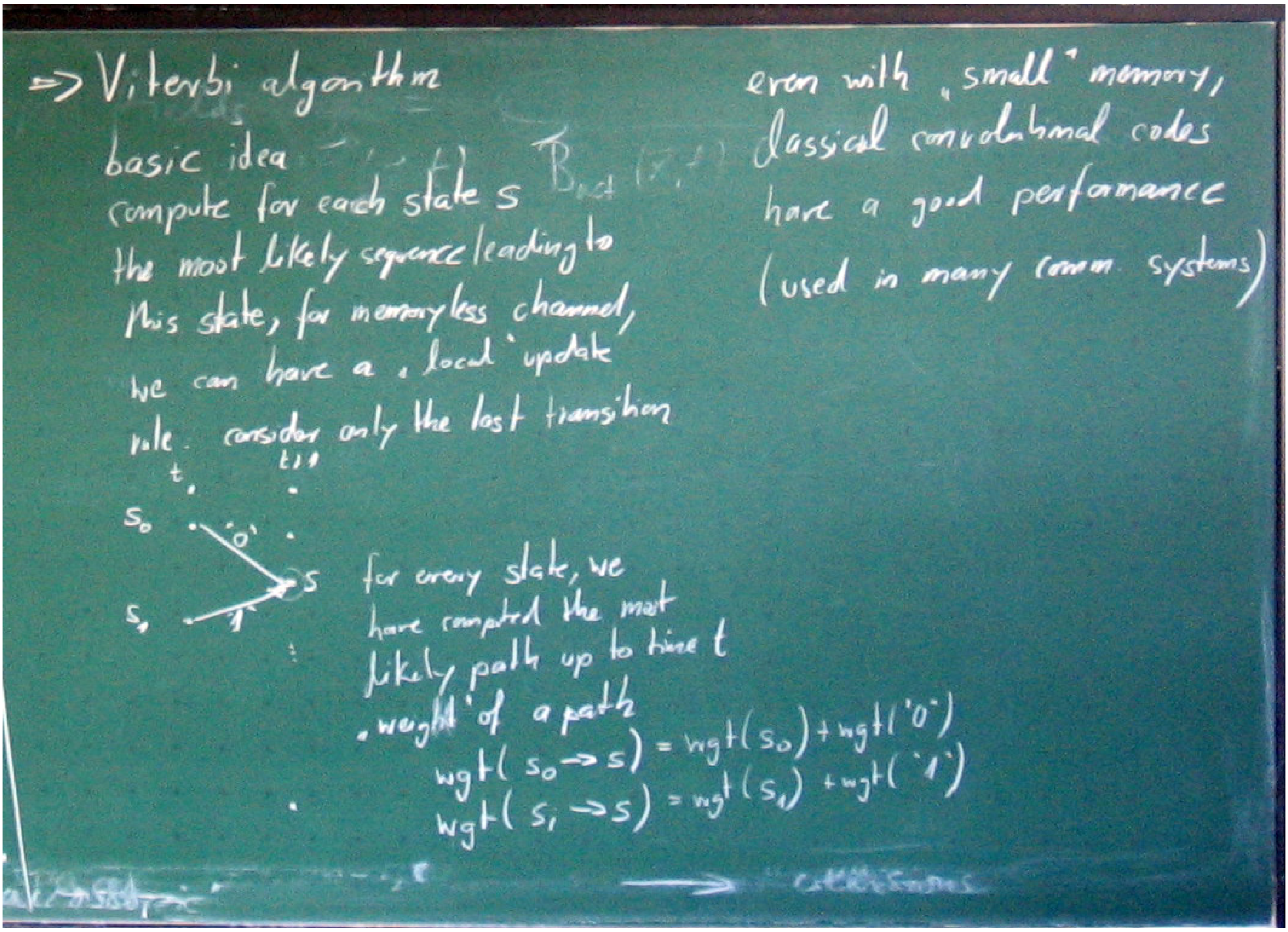
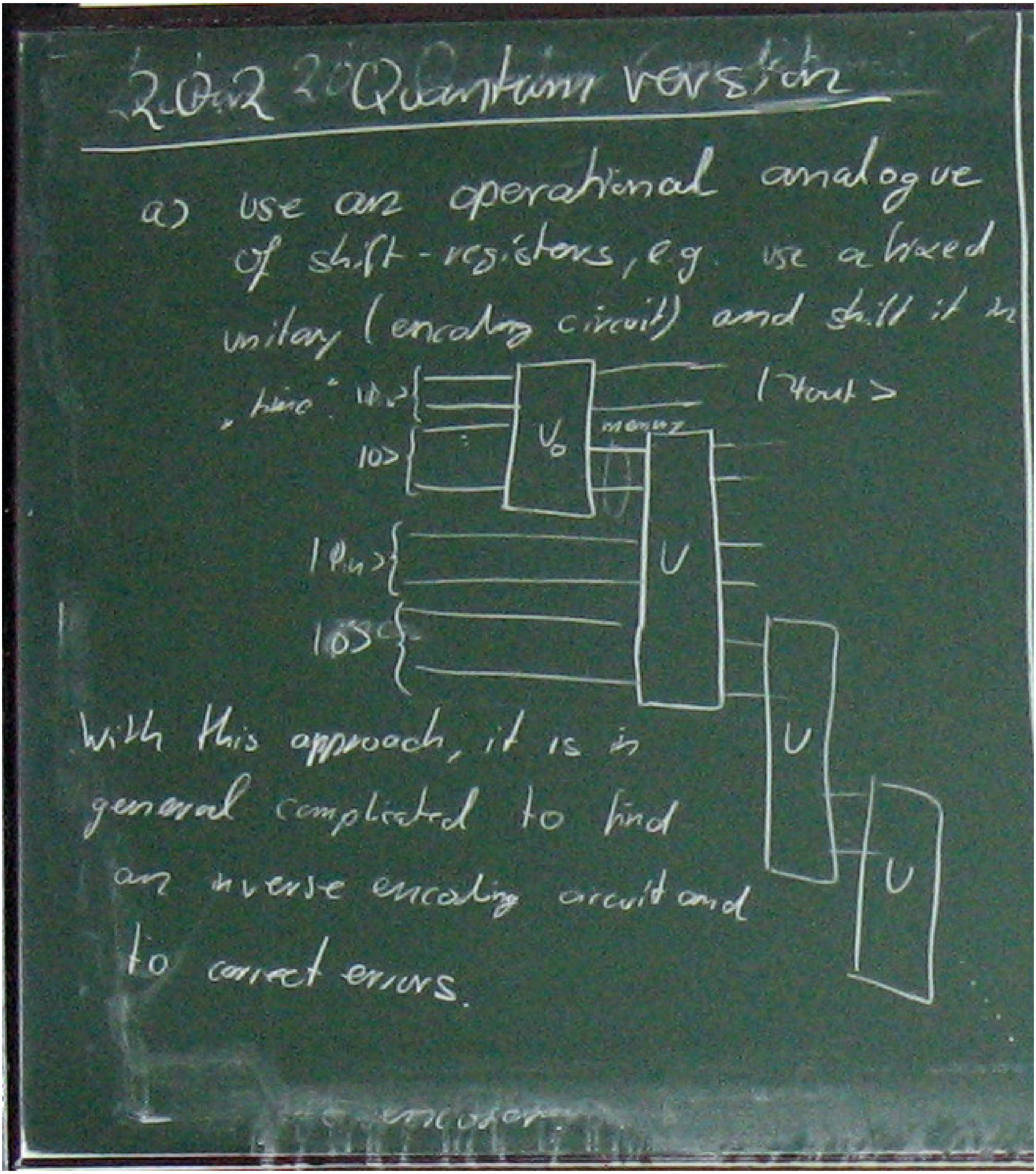
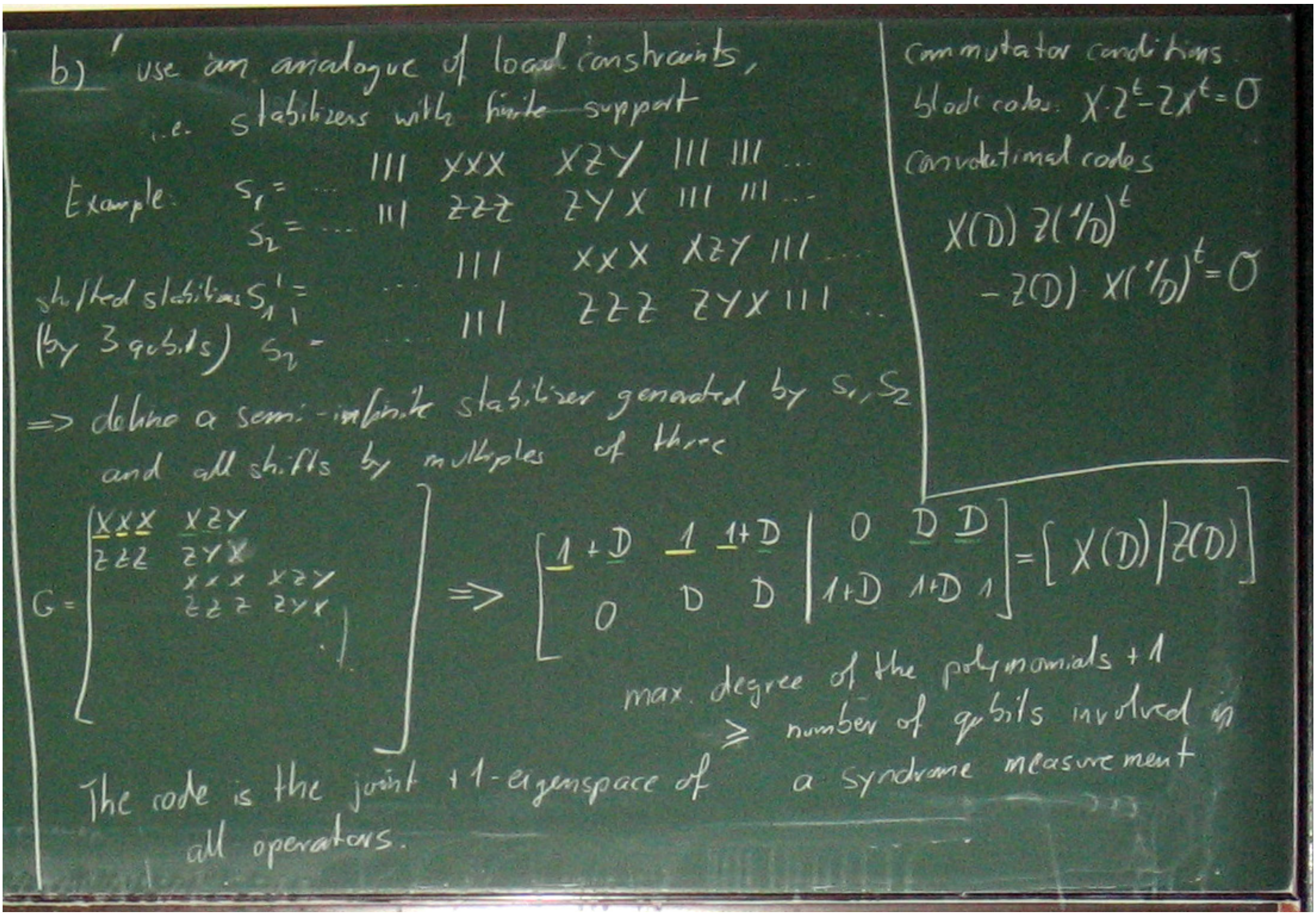

Lecture 20. Quantum Convolutional Codes
20.1 Motivation:
. Block codes encode a fixed number of qubits, good codes have a large blocklength n
good code: $\lim_{n\to\infty} \frac{d}{n} > 0$, $\lim_{n\to\infty} \frac{k}{n} > 0$
⇒ generators of the stabilizer group span many qubits („non-local")
. goals - encode a stream of qubits
- „local" stabilizers
- hopefully still good performance
classical convolutional codes:
linear system with finite memory, i.e. every output symbol depends linearly on the actual input symbol (or a group of inputs) and some previous inputs / the internal state of the encoder


Example: rate R = 1/2
out t,0 = i t + i t+2 + i t+3
inputs
i t+4
i t+3
i t+2
i t
out t,1 = i t+1 + i t+3 + i t+4
linear feed-forward shift-register = polynomial multiplication
feed-back = polynomial division
0 ..... 0 1 0 ... 0
1 1 ... 1 0 ... 0
1/(1+D) = ∑ i≥0 D^i
input 1 produces an infinite sequence
1+D produces the output 1
We use the powers of D as "time indicator", i.e.
an input sequence (a0, a1, ..., aℓ) ≙ a0 + a1 D + a2 D^2 + ... + aℓ D^ℓ


main idea for error correction:
The linear shift-registers produce sequences which obey some linear „local" constraints
=> diagram of the internal state and the output symbols
memory m => 2^m internal states
t-1
t
t+1
0/out0
1/out1
. an edge between states s and s' labelled with the corresponding inputs/outputs
. every codesequence corresponds to path in this trellis diagram
. error correction: finding the closest path given an output sequence
2^m states


=> Viterbi algorithm
basic idea
compute for each state s
the most likely sequence leading to
this state, for memoryless channel,
we can have a "local" update
rule. consider only the last transition
t
t+1
s_0
'0'
s
s_1
'1'
for every state, we
have computed the most
likely path up to time t
"weight" of a path
wgt(s_0 -> s) = wgt(s_0) + wgt('0')
wgt(s_1 -> s) = wgt(s_1) + wgt('1')
even with "small" memory,
classical convolutional codes
have a good performance
(used in many comm. systems)


Quantum version
a) use an operational analogue of shift-registers, e.g. use a fixed unitary (encoding circuit) and shift it in "time"
$|0\rangle$
$|\psi_{out}\rangle$
U_0
memory
$|\varphi_n\rangle$
U
U
U
With this approach, it is in general completed to find an inverse encoding circuit and to correct errors.


b) use an analogue of local constraints,
i.e. stabilizers with finite support
Example: S1 = ... III XXX XZY III III ...
S2 = ... III ZZZ ZYX III III ...
shifted stabilizers S1' = ... III XXX XZY III ...
(by 3 qubits) S2' = ... III ZZZ ZYX III ...
⇒ define a semi-infinite stabilizer generated by S1, S2
and all shifts by multiples of three
G = [XXX XZY; ZZZ ZYX; XXX XZY; ZZZ ZYX; ...]
⇒ [1+D 1 1+D | 0 D D; 0 D D | 1+D 1+D 1] = [X(D) | Z(D)]
max. degree of the polynomials + 1
≥ number of qubits involved in a syndrome measurement
The code is the joint +1-eigenspace of all operators.
Commutator conditions
block codes: X·Z^t − Z X^t = 0
Convolutional codes
X(D) Z(1/D)^t − Z(D) X(1/D)^t = 0

20.3 A bad example

$$G = \begin{bmatrix} ZZ & & & \\ & ZZ & & \\ & & ZZ & \\ & & & \ddots \end{bmatrix} \hat{=} (0 \; 1 \mid 1 + D)$$

↑ does not have a polynomial inverse

⇒ joint +1 eigenstates are

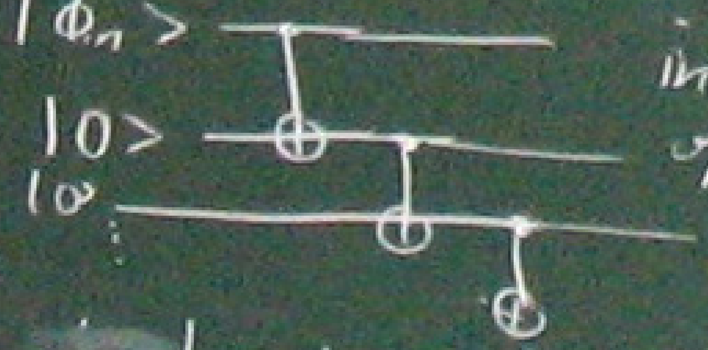
$|\bar{0}\rangle = |00\ldots\rangle$

$|\bar{1}\rangle = |11\ldots\rangle$

in particular: $|\bar{0}\rangle + |\bar{1}\rangle \in \mathcal{C}$

⇒ infinite „cat" state

encoding would be given by



infinite cascade of CNOTs

⇒ try to avoid these infinite depth circuits

20.4 Encoding circuits

based on transforming the stabilizer

$$[\, X(D) \mid Z(D) \,] \rightsquigarrow [\, 0 \mid I \;\; 0 \,]$$

trivial block code
$\hat{=}$ convolutional code without memory

a) local operations $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $P = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

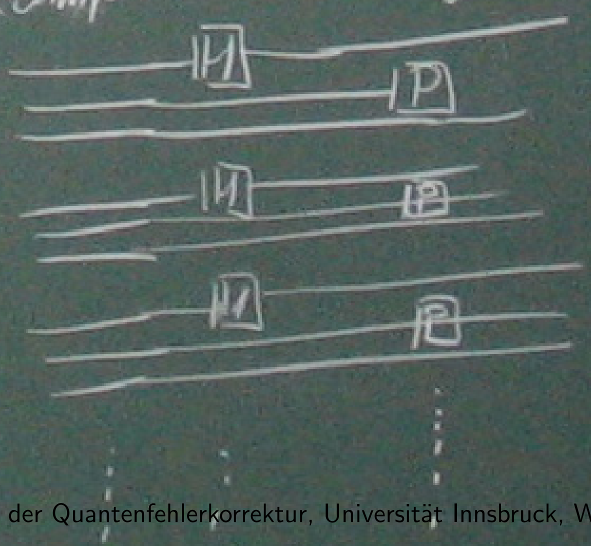
=> operations on the columns of $(X(D) \mid Z(D))$

given by $\tilde{H} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\quad \tilde{P} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

$(X(D) \mid Z(D))$ corresponds to a semi-infinite „block code" matrix

=> apply the gates H or P to <u>all</u> shifted blocks

example: block length 3



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