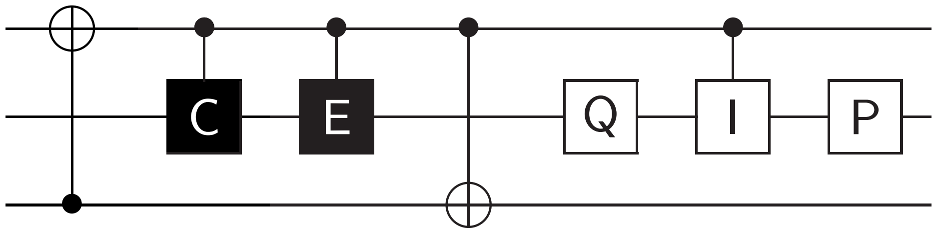
3rd Central European
Quantum Information Processing
Workshop
Znojmo, May 4–8, 2006



Quantum Designs:
MUBs, SICPOVMs, and (a little bit) More

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Partially supported by Landesstiftung Baden-Württemberg gGmbH

Identifying Quantum States

General Problem:

What is the best way to identify an arbitrary unknown quantum state ρ in a d-dimensional Hilbert space?

- ρ is a Hermitian matrix
  $\Longrightarrow d^2 - 1$ real parameters
- one von Neumann measurement provides $d - 1$ independent parameters
  $\Longrightarrow$ at least $d + 1$ different measurements
- general measurements (POVMs)
  $\Longrightarrow$ at least d^2 POVM elements
- goal: "maximal independence" of the measurement results

  $\Longrightarrow$ optimal statistics with no *a priori* knowledge

Mutually Unbiased Bases (MUBs)

- orthogonal bases $\mathcal{B}^j := \{|\psi_k^j\rangle \colon k = 1, \ldots, d\} \subset \mathbb{C}^d$
- basis states are "mutually unbiased":

$$|\langle\psi_k^j|\psi_m^l\rangle|^2 = \begin{cases} 1/d & \text{for } j \neq l, \\ \delta_{k,m} & \text{for } j = l. \end{cases}$$

- at most $d+1$ MUBs in dimension d
- constructions for $d+1$ MUBs only known for prime powers $d = p^e$
- lower bound [Klappenecker & Rötteler, quant-ph/0309120]:

$$N(m \cdot n) \geq \min\{N(m), N(n)\} \geq 3$$

MUBs: Direct Constructions

[see, e.g., Klappenecker & Rötteler, LNCS 2948, 2004, quant-ph/0309120]

Let $d = p^m = q$ be an odd prime power. Then the computational bases and the bases $\mathcal{B}^j := \{|\psi_k^j\rangle : k \in \mathbb{F}_q\}$ for $j \in \mathbb{F}_q$ with

$$|\psi_k^j\rangle := \frac{1}{\sqrt{q}} \sum_{x \in \mathbb{F}_q} \omega_p^{\operatorname{tr}(kx^2+jx)} |x\rangle$$

where $\omega_p := \exp(2\pi i/p)$

are a set of $q+1$ MUBs.

Similar construction for $d = 2^m$ (slightly more technical).

MUBs and Unitary Error Bases

[S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, & F. Vatan, quant-ph/0103162]

Theorem:

There exists k MUBs in dimension d if and only if there are $k(d-1)$ traceless, mutually orthogonal matrices $U_{j,t} \in U(d,\mathbb{C})$ that can be partitioned into k sets of commuting matrices:

$$\mathcal{B} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_k, \qquad \text{where } \mathcal{C}_j \cap \mathcal{C}_l = \emptyset \text{ and } |\mathcal{C}_j| = k-1$$

Each of the k orthogonal bases are the common eigenstates of the commuting matrices in one class $\mathcal{C}_j$.

Ansatz:

Use the matrices $X^a Z^b$ of the Weyl-Heisenberg group.

Weyl-Heisenberg Group

- generators: $$H_d := \langle X, Z \rangle$$

  where $X := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|$

  $$\omega_d := \exp(2\pi i/d)$$

- relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- basis:

$$H_d\big/\zeta(H_d) = \left\{X^a Z^b \colon a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

Jacobi Group (or Clifford Group)

- automorphism group of the Heisenberg group H_d, i.e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of J_d on H_d modulo phases corresponds to the symplectic group $SL(2, \mathbb{Z}_d)$, i.e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix},\ \tilde{T} \in SL(2, \mathbb{Z}_d)$$

- J_d is generated by the discrete Fourier transform and a diagonal matrix "with quadratic phases" (depends on d odd or even)

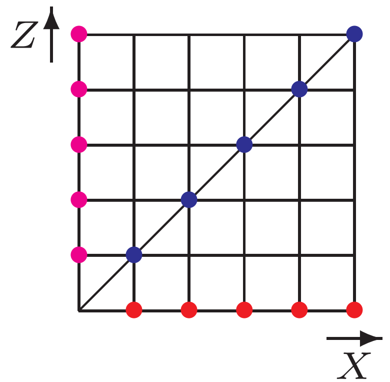
$$P_{\text{even}} = \sum_{i=0}^{d-1} \omega^{i^2/2} |i\rangle\langle i| \qquad P_{\text{odd}} = \sum_{i=0}^{d-1} \omega^{i(i-1)/2} |i\rangle\langle i|$$

Three MUBs in any Dimension

consider the operators

$$\{X^a : a = 1, \ldots, d-1\}, \quad \{Z^a : a = 1, \ldots, d-1\}, \quad \{X^a Z^a : a = 1, \ldots, d-1\}$$

- all matrices are mutually orthogonal, the sets are disjoint, the matrices within each set commute
- geometric picture:



$\Longrightarrow$ the eigenvectors of X, Z, and XZ form three MUBs in any dimension

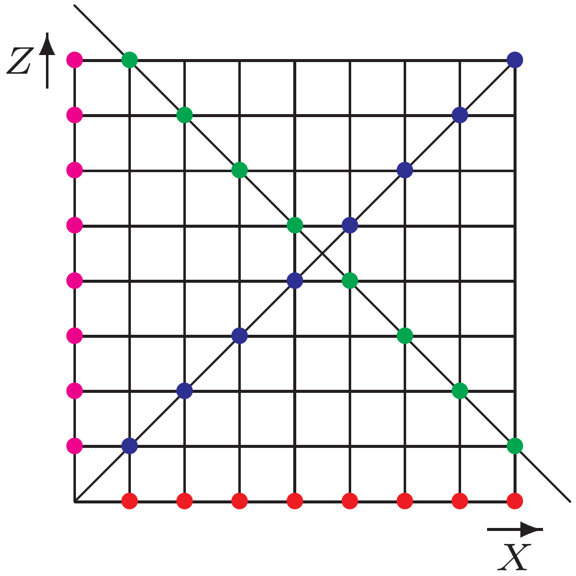
Four MUBs in All Odd Dimensions

Four disjoint sets of operators

$$\{X^a : a = 1, \ldots, d-1\} \qquad \{Z^a : a = 1, \ldots, d-1\}$$

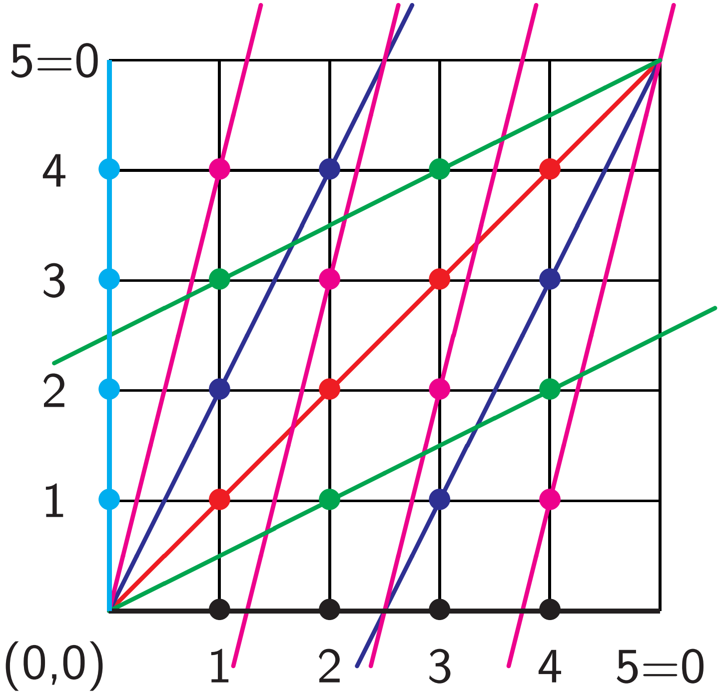
$$\{X^a Z^a : a = 1, \ldots, d-1\} \qquad \{X^a Z^{-a} : a = 1, \ldots, d-1\}$$

geometric picture $(d = 9)$



Finite Geometries

Example: $d = 5$



and all parallel lines

- $d(d+1)$ lines with $d+1$ different slopes on d^2 points
- two non-parallel lines intersect in exactly one point (if d is prime)
- "combinatorial tomography": point values determined by line sums
- constructions for prime powers using finite fields

More than 3 MUBs in Dimension 6?

Ansatz 1:

- start with the eigenvectors of X, Z, and XZ
- search for a vector $|\psi\rangle$ that is unbiased w.r.t. these 18 vectors

$\Longrightarrow$ The system of polynomial equations has no solution.

Ansatz 2:

- start with the eigenvectors of X and Z
- search for a vector $|\psi\rangle$ that is unbiased w.r.t. these 12 vectors
- w.l.o.g, the first coordinate is $1/\sqrt{6}$

$\Longrightarrow$ There are exactly 48 solutions for $|\psi\rangle$.

The 48 Solutions

- Each solution is unbiased with respect to either 4 or 12 other vectors.
- There are 16 subsets of size 6 that are an orthonormal bases.
- No vector is unbiased with respect to one of the 16 bases.

Consequence:
Starting with the eigenvectors of X and Z, we get no more than 3 MUBs in dimension 6.

action of the Jacobi group & geometric interpretation $\Longrightarrow$

Corollary:
Starting with the eigenvectors of “two lines that intersect only in the origin”, we get no more than 3 MUBs in dimension 6.

Unextendable MUBs: Dimension 4

eigenbases of X and Z

$$\mathcal{B}^1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{B}^2 := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

third basis

$$\mathcal{B}^3 := \frac{1}{2}\begin{pmatrix} 1 & e^{ia} & 1 & -e^{ia} \\ 1 & -e^{ia} & 1 & e^{ia} \\ 1 & e^{ib} & -1 & e^{ib} \\ 1 & -e^{ib} & -1 & -e^{ib} \end{pmatrix} \qquad \text{where } a, b \in [0, 2\pi]$$

no additional unbiased vector

MUBs from Mutually Orthogonal Latin Squares

Mutually Orthogonal Latin Squares (MOLS)

- $d \times d$ arrays with entries $1, \ldots, d$
- each row/column contains all entries $1, \ldots, d$
- L^1 and L^2 orthogonal
  $\iff$ given (a, b), $L^1_{i,j} = a$ and $L^2_{i,j} = b$ has exactly one solution (i, j)

Examples

2	3	1
3	1	2
1	2	3

3	1	2
2	3	1
1	2	3

4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2
1	2	3	4	5

3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3
1	2	3	4	5

MUBs from MOLS

The construction of Beth & Wocjan: [quant-ph/0407081]

Given a set of k MOLS of order d, there are $k+2$ MUBs in dimension d^2.

- each latin square defines d incidence vectors of length d^2
- get new vectors by introducing phase factors
- additionally: squares with vertical/horizontal stripes

Example:

Latin square:

2	3	1
3	1	2
1	2	3

incidence vector: (100 010 001)

additional vectors:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & \omega^2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & \omega \end{pmatrix}$$

square | incidence vectors | expanded vectors

2	3	1
3	1	2
1	2	3

(100010001)

(010001100)

(001100001)

$$\begin{pmatrix}1&0&0&0&1&0&0&0&1\end{pmatrix}$$
$$\begin{pmatrix}1&0&0&0&\omega&0&0&0&\omega^2\end{pmatrix}$$
$$\begin{pmatrix}1&0&0&0&\omega^2&0&0&0&\omega\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0&0&1&1&0&0\end{pmatrix}$$
$$\begin{pmatrix}0&1&0&0&0&\omega&\omega&0&0\end{pmatrix}$$
$$\begin{pmatrix}0&1&0&0&0&\omega&\omega^2&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&1&0&0&0&0&1\end{pmatrix}$$
$$\begin{pmatrix}0&0&1&\omega&0&0&0&0&\omega\end{pmatrix}$$
$$\begin{pmatrix}0&0&1&\omega^2&0&0&0&0&\omega^2\end{pmatrix}$$

3	3	3
2	2	2
1	1	1

(111000000)

(000111000)

(000000111)

$$\begin{pmatrix}1&1&1&0&0&0&0&0&0\end{pmatrix}$$
$$\begin{pmatrix}1&\omega&\omega^2&0&0&0&0&0&0\end{pmatrix}$$
$$\begin{pmatrix}1&\omega^2&\omega&0&0&0&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&1&1&1&0&0&0\end{pmatrix}$$
$$\begin{pmatrix}0&0&0&1&\omega&\omega&0&0&0\end{pmatrix}$$
$$\begin{pmatrix}0&0&0&1&\omega&\omega^2&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0&0&0&1&1&1\end{pmatrix}$$
$$\begin{pmatrix}0&0&0&0&0&0&1&\omega&\omega\end{pmatrix}$$
$$\begin{pmatrix}0&0&0&0&0&0&1&\omega^2&\omega^2\end{pmatrix}$$

Symmetric Informationally Complete POVMs

[J. M. Renes, R. Blume-Kohout, A. J. Scott, & C. M. Caves, quant-ph/0310075]

- POVM with d^2 rank-one elements $E_j = \Pi_j/d$ with $\Pi_j = |\phi_j\rangle\langle\phi_j|$
- The d^2 elements form a basis of $\mathbb{C}^{d\times d}$.
  $\Longrightarrow$ "informationally complete"
- expectation values $p_j = \mathrm{tr}(\rho E_j)$ "maximally independent":

$$\mathrm{tr}\left(\Pi_j \Pi_k\right) = |\langle\phi_j|\phi_k\rangle|^2 = \frac{1}{d+1} \qquad \text{for } j \neq k,$$

  $\Longrightarrow$ "symmetric"
- algebraic solutions for dimension $d = 2, 3, 4, 5, 8$ [e.g. Zauner 99]
- numerical solutions for $d \leq 45$ [Renes et al. 03]

Zauner's Conjecture

[G. Zauner, Dissertation, Universität Wien, 1999]

Conjecture:

For every dimension $d \geq 2$ there exists a SIC-POVM whose elements are the orbit of a rank-one operator E_0 under the Heisenberg group H_d.
What is more, E_0 commutes with an element T of the Jacobi group J_d.
The action of T on H_d modulo the center has order three.

support for this conjecture:

- algebraic solutions by [Zauner, Appleby] for $d = 2, 3, 4, 5, 7, 19$ (only prime powers)
- numerical evidence by [Renes et al.] for $d \leq 45$

SIC-POVM in Dimension 6

Ansatz 1:

SIC-POVM that is the orbit under H_d, i.e.,

$$|\phi_{a,b}\rangle := X^a Z^b |\phi_0\rangle \qquad (1)$$

$$|\langle\phi_{a,b}|\phi_{a',b'}\rangle|^2 = \begin{cases} 1 & \text{for } (a,b)=(a',b'), \\ 1/(d+1) & \text{for } (a,b)\neq(a',b') \end{cases} \qquad (2)$$

$$|\phi_0\rangle = \sum_{j=0}^{d-1}(x_{2j}+ix_{2j+1})|b_j\rangle,$$

($x_0,\ldots,x_{2d-1}$ are real variables, $x_1=0$)

$\Longrightarrow$ polynomial equations for $2d-1$ variables, but too complicated for $d=6$

SIC-POVM in Dimension 6 (cntd.)

Ansatz 2:

SIC-POVM that is the orbit under H_d,
additionally: $|\phi_0\rangle$ lies in a (degenerate) ℓ-dimensional eigenspace of some $T \in J_d$

$$|\phi_0\rangle = \sum_{j=0}^{\ell-1} (x_{2j} + i x_{2j+1}) |b_j\rangle,$$

here: $\ell = 3$, i.e., only 5 variables

$\Longrightarrow$ algebraic solution computed using Magma

- 144 complex solutions for the real variables
  $\Longrightarrow$ only the real solutions are valid
- in total 96 “different” such SIC-POVMs, but all these SIC-POVMs are related by complex conjugation or a global basis change

SIC-POVM for $d = 6$

$$|\phi_0\rangle = \sum_{i=1}^{6} v_i |i\rangle$$

$$\begin{aligned}
v_1 := \Big(& \big(336(\sqrt{7}-\sqrt{21})\theta_1 - 42\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} - 378\big)\theta_2^2 \\
& + \big(56(3\sqrt{7}-2\sqrt{3}+3)\theta_1 + 3\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} + 63\big)\theta_2 \\
& + (168 - 24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7})\theta_1 + 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} - 6\Big)i \\
& + \big(336(\sqrt{7}+\sqrt{21})\theta_1 + 42\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} + 378\big)\theta_2^2 \\
& + \big(56(3\sqrt{7}-2\sqrt{3}-3)\theta_1 - 3\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} - 63\big)\theta_2 \\
& + (24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7} - 168)\theta_1 - 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} + 6,
\end{aligned}$$

$$\begin{aligned}
v_2 := \Big(& \big(672(\sqrt{7}-\sqrt{21})\theta_1 - 168\sqrt{3} + 504\big)\theta_2^2 \\
& + \big(28(3\sqrt{21}+5\sqrt{3}-3\sqrt{7}-15)\theta_1 - 42\sqrt{3} + 126\big)\theta_2 \\
& + (336 - 48\sqrt{21} - 112\sqrt{3} + 48\sqrt{7})\theta_1 - 12\sqrt{21} - 12\sqrt{3} + 12\sqrt{7} + 36\Big)i \\
& - (84\sqrt{21} - 252\sqrt{3} - 252\sqrt{7} + 252)\theta_2^2 \\
& + (84(\sqrt{21}+\sqrt{3}-3\sqrt{7}-1)\theta_1 - 6\sqrt{21} + 18\sqrt{7})\theta_2 - 24\sqrt{3} + 24,
\end{aligned}$$

$$v_3 := 6(\sqrt{7}-\sqrt{3})i + 6\sqrt{21} + 12\sqrt{3} - 12\sqrt{7} - 18$$

SIC-POVM for $d = 6$ (cntd.)

$$
\begin{aligned}
v_4 := \Big(& \big(336(\sqrt{7}-\sqrt{21})\theta_1 + 126\sqrt{21} - 42\sqrt{3} - 126\sqrt{7} + 126\big)\theta_2^2 \\
& + \big(56(6 - 3\sqrt{21} - 2\sqrt{3} + 3\sqrt{7})\theta_1 - 9\sqrt{21} - 21\sqrt{3} + 9\sqrt{7} + 63\big)\theta_2 \\
& + ((168 - 24\sqrt{21} - 56\sqrt{3} + 24\sqrt{7})\theta_1 + 6\sqrt{21} + 18\sqrt{3} - 6\sqrt{7} - 54)\Big) i \\
& + \big(336(\sqrt{21} - 3\sqrt{7})\theta_1 + 42\sqrt{21} - 378\sqrt{3} - 126\sqrt{7} + 378\big)\theta_2^2 \\
& + \big(168(\sqrt{3} - 1)\theta_1 - 3\sqrt{21} + 63\sqrt{3} + 9\sqrt{7} - 63\big)\theta_2 \\
& + (24\sqrt{21} + 168\sqrt{3} - 72\sqrt{7} - 168)\theta_1 + 6 - 6\sqrt{21} - 6\sqrt{3} + 18\sqrt{7}, \\
v_5 := \Big(& \big(672\sqrt{7}\theta_1 + 84\sqrt{21} - 168\sqrt{3} + 252\big)\theta_2^2 \\
& - \big((84\sqrt{21} - 140\sqrt{3} + 84\sqrt{7} - 84)\theta_1 - 6\sqrt{21} + 42\sqrt{3}\big)\theta_2 \\
& - (112\sqrt{3}\theta_1 - 48\sqrt{7}\theta_1 + 12\sqrt{3} - 12\sqrt{7} + 24)\Big) i \\
& + (672\sqrt{7}\theta_1 - 84\sqrt{21} - 168\sqrt{3} - 252)\theta_2^2 \\
& + \big((84\sqrt{21} + 140\sqrt{3} - 84\sqrt{7} - 84)\theta_1 - 6\sqrt{21} - 42\sqrt{3}\big)\theta_2 \\
& - 112\sqrt{3}\theta_1 + 48\sqrt{7}\theta_1 - 12\sqrt{3} + 12\sqrt{7} + 24, \\
v_6 := & \, 6(\sqrt{7} - \sqrt{3})i - 6\sqrt{21} + 18.
\end{aligned}
$$

(Weak) Analogies to Finite Geometry

[W. K. Wootters, quant-ph/0406032]

affine planes	MUBs	SIC-POVMs
d^2 points	d^2 *Wigner operators*	d^2 states
$d(d+1)$ lines	$d(d+1)$ states	$d(d+1)$ operators "B_α"
lines are • parallel, or • intersect in one point	states are • orthogonal • unbiased	different cases for trace inner products of B_α and B_β
only known to exist for prime powers	constructions for prime powers	*conjectured:* *all dimensions*

Why Dimension 6?

- $6 = 2 \cdot 3$ is the smallest non-prime power.
- There is no affine plane of order six.
- There are no two mutually orthogonal Latin squares (MOLS).
  $\Longrightarrow$ This *could* imply that there a no more then 3 MUBs.
- There are no more than 3 MUBs that are related to the Weyl-Heisenberg group respectively $\mathbb{Z}_6 \times \mathbb{Z}_6$.

but: A SIC-POVM in dimension 6 exists!

General Framework: Quantum Designs

[G. Zauner, Dissertation, Universität Wien, 1999]

Set of projection operators P_i with certain regularities:

object	**classical**	**quantum**
block	subset $B_i \subseteq \{1, \ldots, n\}$	projection P_i
$\|B_i\| \mathrel{\hat{=}} \mathrm{tr}(P_i)$	constant	regular/not regular
intersection	$\|B_i \cap B_j\|$	$\mathrm{tr}(P_i P_j)$
degree	$\#\{\|B_i \cap B_j\| : i \neq j\}$	$\#\{\mathrm{tr}(P_i P_j) : i \neq j\}$
partitioning	resolvable	$\approx$ set of observables

Theory allows to derive bounds on these quantities, e. g. $\mathrm{tr}(P_i P_j) =: \lambda(d, m)$

$d \backslash m$	2	3	4	5	6	7	8	9
2	0	$\frac{1}{4}$	$\frac{1}{3}$					
3	0	0	$\frac{1}{9}$	–	$\frac{1}{5}$	$\frac{2}{9}$	$\frac{5}{21}$	$\frac{1}{4}$

Conclusion/Outlook

- Proof that there exist SIC-POVMs in (some) non-prime-power dimensions. (Also algebraic solutions for $d = 2, \ldots, 15$, but not yet for $d = 14$.)
- The connection to finite geometry is too weak to make any statement about the of more than 3 MUBs in dimension 6.
- results on "unextendable" MUBs for $d = 4, 6$
- MUBs from MOLS
  $\Longrightarrow$ at least $d^{14.8} + 2$ MUBS in dimension d^2 for $d \geq d_0$

open problems:

- Are there more than 3 MUBs in dimension 6?
- General construction for SIC-POVMs.

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