Computational Algebra Seminar
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Quantum Error Correction
– Discrete Math. Meets Physics

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Quantum Information Processing

Main Idea

Computation based on the laws of quantum mechanics

Main Algorithms (so far)

- integer factorisation, discrete log over $\mathbb{F}_p^*$ [Shor]
  generalisation to other groups (Abelian Hidden Subgroup Problems (HSP))
  $\Longrightarrow$ exponential speed-up
- quantum "searching" [Grover]
  more precise: find the solutions of $f(x) = 1$ for a (efficiently computable) function $f : M \longrightarrow \{0, 1\} \Longrightarrow$ quadratic speed-up

Main Problem

Quantum mechanical systems are easy to disturb.

Quantum Systems

Model

- system is modelled by a complex Hilbert space $\mathcal{H}$
  in our context: finite dimensional $\mathcal{H} \cong \mathbb{C}^d$
- composed systems are modelled by the tensor product of the component spaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \Longrightarrow$ exponential growth of dimension

Pure Quantum State

- normalised vector in $\mathcal{H}$
- basis states: $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ ("classical information")
- **superposition**:

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \quad \text{where} \sum_{i=0}^{d-1} |\alpha_i|^2 = 1$$

Quantum Operations

Unitary Operations

- all unitary operations on $\mathcal{H}$ are valid
- local operations: tensor product of $U \in \mathcal{U}(d)$ with identity matrices for the other tensor components

Measurements

- *observable* A: Hermitian matrix
- spectral decomposition of A yields (real) eigenvalues λ_i and orthogonal projections P_i onto the corresponding eigenspaces
- *measurement result* λ_i:
  – random value with probability $p_i := \langle\psi| P_i |\psi\rangle$
  – projection (and re-normalisation): $|\psi'\rangle = \frac{1}{\sqrt{p_i}} P_i |\psi\rangle$

Entanglement

- superposition of basis states need not be tensor products, e. g.,

$$\left|\Phi^+\right\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle_L \otimes |0\rangle_R + |1\rangle_L \otimes |1\rangle_R\right)$$

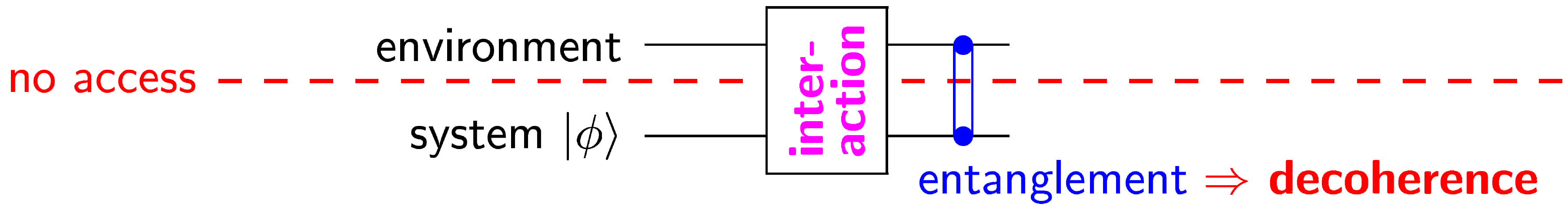
- measuring $\sigma_z := \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ with eigenstates $|0\rangle$ and $|1\rangle$
  - measuring $\sigma_z \otimes I$ yields with prob. $1/2$ either

$$|0\rangle_L \otimes |0\rangle_R \quad \text{or} \quad |1\rangle_L \otimes |1\rangle_R$$

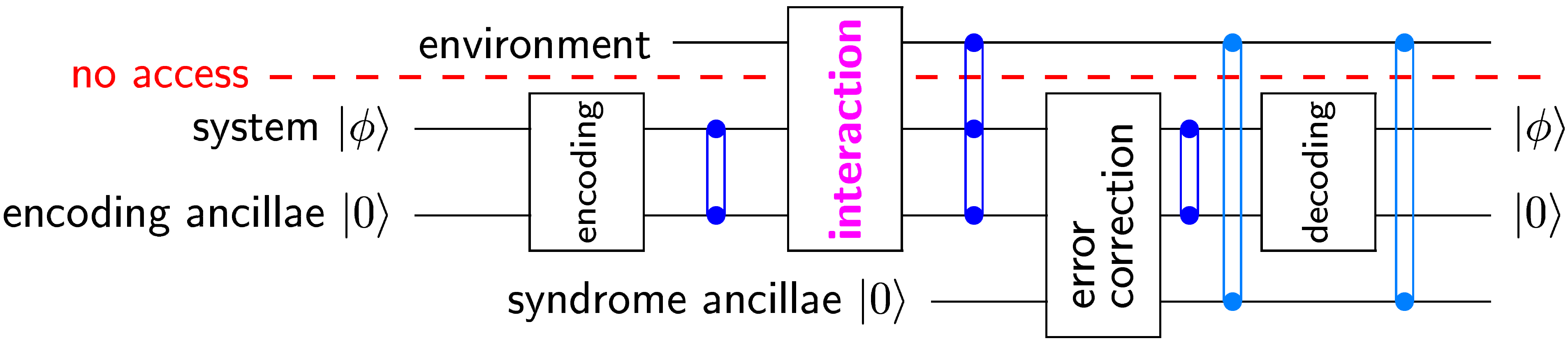
  - then measuring $I \otimes \sigma_z$
    * gives a deterministic results, if the outcome of the first measurement is known
    * is completely random (prob. 1/2), if the first outcome is not known

System & Environment

Without Error Correction

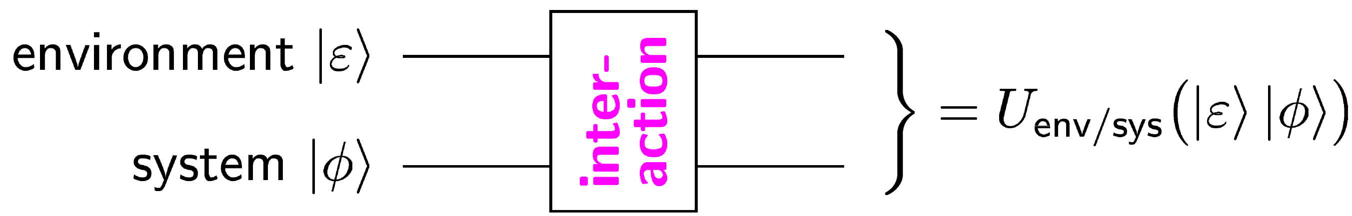


With Error Correction



Interaction System/Environment

“Closed” System



“Channel”

$$\mathsf{Q}\colon \rho_{\text{in}} := |\phi\rangle\langle\phi| \longmapsto \rho_{\text{out}} := \mathsf{Q}(|\phi\rangle\langle\phi|) := \sum_i E_i \rho_{\text{in}} E_i^\dagger$$

with error operators (Kraus operators) E_i

Local/low correlated errors

- product channel $\mathsf{Q}^{\otimes n}$ where Q is “close” to identity
- Q can be expressed (approximated) with error operators $\tilde{E}_i$ such that each E_i acts on few subsystems

QECCs for Local Error Models

Quantum Error-Correcting Code (QECC)

$$\mathcal{C} \subseteq (\mathbb{C}^q)^{\otimes n} \quad \text{where } \dim \mathcal{C} = q^k$$

Notation $\mathcal{C} = [\![n, k, d]\!]_q$

n: number of subsystems used in total

k: number of (logical) subsystems encoded

d: "minimum distance"

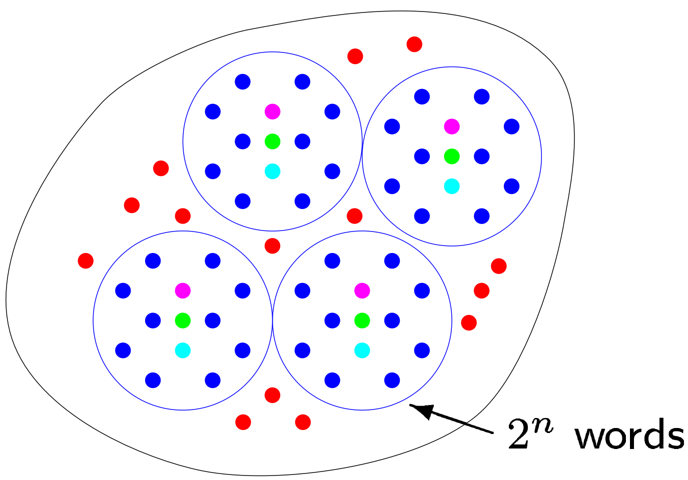
- correct all errors acting on at most $(d-1)/2$ subsystems
- detect all errors acting on less than d subsystems

Basic Ideas

Partitioning all (binary) words

– combinatorics

– (linear) algebra

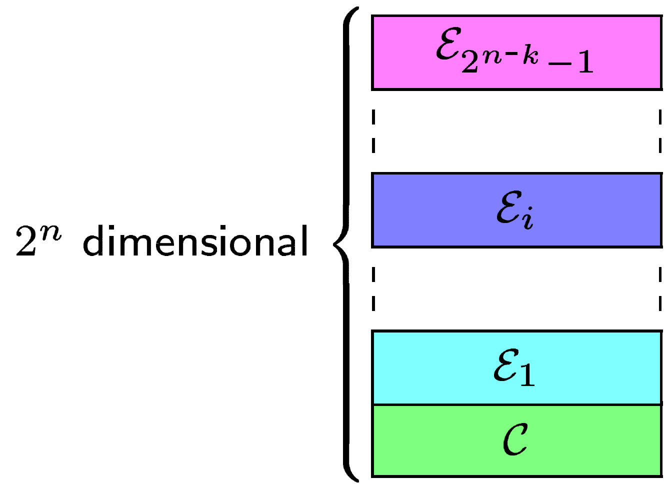


codewords

bounded weight errors

other errors

orthogonal decomposition



$$(\mathbb{C}^d)^{\otimes n} = \mathcal{H}_{\mathcal{C}} \oplus \mathcal{H}_{\mathcal{E}_1} \oplus \ldots \oplus \mathcal{H}_{\mathcal{E}_i} \oplus \ldots$$

Characterisation of QECCs

[E. Knill & R. Laflamme, PRA **55**, 900–911 (1997)]

A subspace $\mathcal{C}$ of $\mathcal{H}$ with orthonormal basis $\{|c_1\rangle, \ldots, |c_K\rangle\}$ is an error-correcting code for the error operators $\mathcal{E} = \{E_1, \ldots, E_\mu\}$, if there exists constants $\alpha_{k,l} \in \mathbb{C}$ such that for all $|c_i\rangle$, $|c_j\rangle$ and for all $E_k, E_l \in \mathcal{E}$:

$$\langle c_i| E_k^\dagger E_l |c_j\rangle = \delta_{i,j}\alpha_{k,l}. \qquad (1)$$

It is sufficient that (1) holds for a vector space basis of $\mathcal{E}$.

$\Longrightarrow$ only a finite set of errors

Error Basis

Unitary Error Basis

set of d^2 unitary matrices that forms a vector space basis of all $d \times d$ matrices

Nice Unitary Error Basis

basis elements U_g are labelled by group elements $g \in G$ with the property:

$$U_g U_h = \omega(g, h) U_{g*h}$$

$\Longrightarrow$ irreducible projective representations

[E. Knill, Group Representations, Error Bases and Quantum Codes, quant-ph/9608049 (1996)]

[A. Klappenecker & M. Rötteler, Beyond Stabilizer Codes I: Nice Error Bases, IEEE Transactions on Information Theory, 48(8), pp. 2392–2395, (2002)]

Heisenberg-Weyl Group

Shift & Phase Operators

for arbitrary dimension d with basis $B := \{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$

- shift operator $X\colon\ |x\rangle \mapsto |(x+1) \bmod d\rangle$
- phase operator $Z\colon\ |x\rangle \mapsto \omega_d^x |x\rangle$ where $\omega_d := \exp 2\pi i/d$

Heisenberg-Weyl Group:

$$G := \langle X, Z \rangle$$

order	$\lvert G \rvert = d^3$
centre	$\zeta(G) = \langle \omega_d I \rangle$
quotient	$G/\zeta(G) \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$

Qudits and Finite Fields

Qudits

- tensor product of quantum systems of dimension d,
  in particular $d = p^m$, p prime
- *single qudit*

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i \, |i\rangle \qquad \text{where } \alpha_i \in \mathbb{C} \text{ and } \sum_{i=0}^{d-1} |\alpha_i|^2 = 1$$

  labels i of the basis states from an arbitrary set $\mathcal{A}$ with d elements, e. g.

$$\mathcal{A} = \{0, 1, \ldots, d-1\} \text{ or } \mathcal{A} = \mathbb{F}_{p^m} \text{ for } d = p^m,\ p \text{ prime}$$

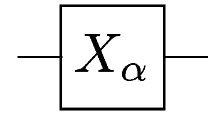
Finite Fields

- trace: $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ where $\mathrm{tr}(\alpha) := \sum_{i=0}^{m-1} \alpha^{p^i} \in \mathbb{F}_p$
- in $\mathbb{F}_q$ there exists a primitive $(q-1)$th root of unity

Single Qudit Gates

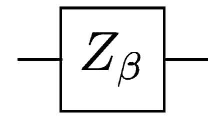
- $X_\alpha := \sum_{x \in \mathbb{F}_q} |x+\alpha\rangle\langle x|$ for $\alpha \in \mathbb{F}_q$

  $= X_{\alpha_1} \otimes X_{\alpha_2} \otimes \ldots \otimes X_{\alpha_m}$

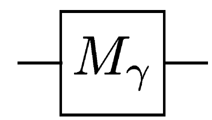


- $Z_\beta := \sum_{z \in \mathbb{F}_q} \omega^{\mathrm{tr}(\beta z)} |z\rangle\langle z|$ for $\beta \in \mathbb{F}_q$ $(\omega := \omega_p = \exp(2\pi i/p))$

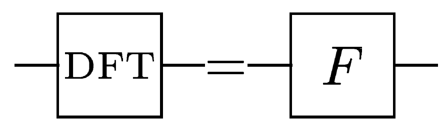
  $= Z_{\beta_1} \otimes Z_{\beta_2} \otimes \ldots \otimes Z_{\beta_m}$



- $M_\gamma := \sum_{y \in \mathbb{F}_q} |\gamma y\rangle\langle y|$ for $\gamma \in \mathbb{F}_q \setminus \{0\}$

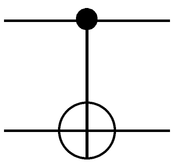


- $\mathrm{DFT} := \frac{1}{\sqrt{q}} \sum_{x,z \in \mathbb{F}_q} \omega^{\mathrm{tr}(xz)} |z\rangle\langle x|$

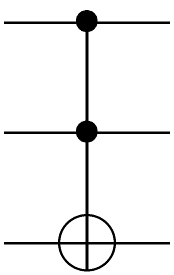


Universal Gates

- $\mathrm{ADD}^{(1,2)} := \sum_{x,y\in\mathbb{F}_q} |x\rangle_1 \, |x+y\rangle_2 \, \langle y|_2 \, \langle x|_1$



- $\mathrm{HORNER}^{(1,2,3)} := \sum_{a,x,b\in\mathbb{F}_q} |a\rangle_1 \, |x\rangle_2 \, |ax+b\rangle_3 \, \langle b|_3 \, \langle x|_2 \, \langle a|_1$



$\Longrightarrow$ any (classical) reversible function

$$f\colon \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$$

can be implemented with the HORNER-gate (using ancillae)

(Non-binary) Quantum Codes (QECCs)

Error Basis for Qudits

[A. Ashikhmin & E. Knill, IEEE-IT **47**, 3065–3072 (2001)]

$$\mathcal{E} = \{X_\alpha Z_\beta \colon \alpha, \beta \in \mathbb{F}_q\}.$$

commutator relations:

$$X_\alpha Z_\beta = \omega^{-\operatorname{tr}(\alpha\beta)} Z_\beta X_\alpha$$

and

$$(X_\alpha Z_\beta)(X_{\alpha'} Z_{\beta'}) = \omega^{\operatorname{tr}(\alpha'\beta - \alpha\beta')}(X_{\alpha'} Z_{\beta'})(X_\alpha Z_\beta)$$

Stabiliser Code

$\mathcal{C}$ is an eigenspace of $\mathcal{S}$ w.r.t. some irred. (projective) character χ

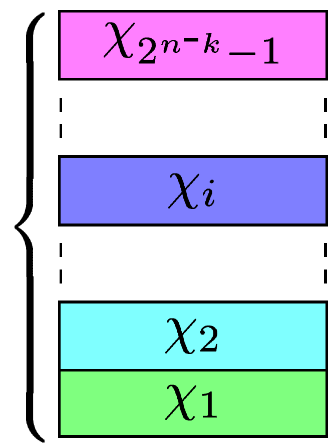
where the *stabiliser* $\mathcal{S}$ is an Abelian subgroup of $\mathcal{E}^{\otimes n}$

Stabiliser Codes

Representation Theory

$\mathcal{C}$ is an eigenspace of $\mathcal{S}$ w.r.t. some irred. character χ_1

decomp. into
irred. components



the orthogonal spaces are labelled by the character (values)

$\Longrightarrow$ operations that change the character (value) can be detected

Stabiliser Codes (contd.)

Error Group

$$\mathcal{G}_1 := \langle X_\alpha, Z_\beta \colon \alpha, \beta \in \mathbb{F}_q \rangle, \qquad |\mathcal{G}_1| = pq^2, \qquad \text{centre } \zeta(\mathcal{G}_1) = \langle \omega I \rangle$$

unique representation of the elements of $\mathcal{G}_1$:

$$\omega^\gamma X_\alpha Z_\beta \qquad \text{where } \gamma \in \mathbb{F}_p = \{0, \ldots, p-1\} \text{ and } \alpha, \beta \in \mathbb{F}_q$$

n qudits:

$$\mathcal{G}_n := \mathcal{G}^{\otimes n}, \qquad |\mathcal{G}_n| = pq^{2n}, \qquad \text{centre } \zeta(\mathcal{G}_n) = \langle \omega I \rangle$$

unique representation of the elements of $\mathcal{G}_n$:

$$\omega^\gamma (X_{\alpha_1} Z_{\beta_1}) \otimes (X_{\alpha_2} Z_{\beta_2}) \otimes \ldots \otimes (X_{\alpha_n} Z_{\beta_n}) =: \omega^\gamma X_{\boldsymbol{\alpha}} Z_{\boldsymbol{\beta}}$$

where $\gamma \in \mathbb{F}_p$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_q^n$

quotient group:

$$\overline{\mathcal{G}}_n := \mathcal{G}_n / \langle \omega I \rangle \cong (\mathbb{F}_q \times F_q)^n \cong \mathbb{F}_q^n \times \mathbb{F}_q^n$$

Stabiliser Codes (contd.)

Abelian Subgroups $\mathcal{S}$ of $\mathcal{G}_n$

symplectic inner product on $\mathbb{F}_q^n \times \mathbb{F}_q^n$:

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) * (\boldsymbol{\alpha}', \boldsymbol{\beta}') := \sum_{i=1}^{n} \operatorname{tr}(\alpha_i' \beta_i - \alpha_i \beta_i') \tag{2}$$

$C \subseteq \mathbb{F}_q^n \times \mathbb{F}_q^n$ self-orthogonal

$$:\Longleftrightarrow C \subseteq C^* := \{\boldsymbol{d} \colon \boldsymbol{d} \in \mathbb{F}_q^n \times \mathbb{F}_q^n | \forall \boldsymbol{c} \in \mathcal{C} \colon \boldsymbol{d} * \boldsymbol{c} = 0\}$$

commutator relations in $\mathcal{G}_n$:

$$(X_{\boldsymbol{\alpha}} Z_{\boldsymbol{\beta}})(X_{\boldsymbol{\alpha}'} Z_{\boldsymbol{\beta}'}) = \omega^{(\boldsymbol{\alpha}, \boldsymbol{\beta}) * (\boldsymbol{\alpha}', \boldsymbol{\beta}')} (X_{\boldsymbol{\alpha}'} Z_{\boldsymbol{\beta}'})(X_{\boldsymbol{\alpha}} Z_{\boldsymbol{\beta}})$$

$\mathcal{S}$ Abelian subgroup

$\Longleftrightarrow (\boldsymbol{\alpha}, \boldsymbol{\beta}) * (\boldsymbol{\alpha}', \boldsymbol{\beta}') = 0$ for all $\omega^{\gamma}(X_{\boldsymbol{\alpha}} Z_{\boldsymbol{\beta}})$, $\omega^{\gamma'}(X_{\boldsymbol{\alpha}'} Z_{\boldsymbol{\beta}'}) \in \mathcal{S}$

Stabiliser Codes (contd.)

Classical Error-Correcting Codes

Abelian subgroups $\mathcal{S}$ of $\mathcal{G}_n$ correspond to additively closed self-orthogonal subsets C of $\mathbb{F}_q^n \times \mathbb{F}_q^n$.

$\Longrightarrow$ $\mathbb{F}_p$-linear codes over $\mathbb{F}_q = \mathbb{F}_{p^m}$

Variations (stronger conditions)

- $\mathbb{F}_q$-linear codes over $\mathbb{F}_q \times \mathbb{F}_q \cong \mathbb{F}_{q^2}$

  inner product: $$(\boldsymbol{\alpha}, \boldsymbol{\beta}) * (\boldsymbol{\alpha}', \boldsymbol{\beta}') := \sum_{i=1}^{n} \alpha_i' \beta_i - \alpha_i \beta_i'$$

- $\mathbb{F}_{q^2}$-linear codes over $\mathbb{F}_{q^2}$

  Hermitian inner product: $$\boldsymbol{v}' * \boldsymbol{w} = \sum_{i=1}^{n} v_i w_i^q$$

Effect of Errors

operation $E \in \mathcal{G}_n$	vector $\boldsymbol{v}$	effect
operations in the stabiliser $\mathcal{S}$	elements of C	no effect
operations in the normaliser $\mathcal{N}$ of $\mathcal{S}$ in $\mathcal{G}_n$	proper cosets of C in C^*	preserve the code space
operations that change the character χ_0	proper cosets of C^*	leave the code space

Minimum Distance

$$d_{\mathsf{min}} := \min\{\mathrm{wgt}(\boldsymbol{c})\colon \boldsymbol{c} \in C^* \setminus C\}$$

Outlook

Results

- quantum error-correction is possible
- QECCs allow "encoded" operations
  $\Longrightarrow$ fault tolerant quantum computation:
  arbitrary long quantum computations can be stabilised with only poly. overhead if all components are not too bad ($p_{\text{error}} < 10^{-6}$–$10^{-4}$)
- codes for $q > 2$ have better parameters

Challenge:

Is there a QECC $\mathcal{C} = [\![7, 1, 4]\!]_4$?

partial result: There is no such code which is $\mathbb{F}_{16}$-linear.