WQMNR

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Workshop On Quantum Marginals and Numerical Ranges

Entanglement Polytopes of Some Five Qubit States

Markus Grassl

Markus.Grassl@mpl.mpg.de

www.codetables.de

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FAU
FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE
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Overview

- Local Spectra
- SLOCC Equivalence
- Local Invariants & Covariants
- Entanglement Polytopes & Covariants
- Computing Covariants & Entanglement Polytopes
- Three Qubits
- Four Qubits
- Five Qubits (work in progress)
- Summary & Outlook

Local Spectra

Given a pure state of n particles

$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_n} x_{i_1,i_2,\ldots,i_n} |i_1\rangle|i_2\rangle\ldots|i_n\rangle \in \mathbb{C}^{d_1}\otimes\ldots\otimes\mathbb{C}^{d_n}$$

One-particle reduced density matrices

$$\rho_1 = \mathrm{tr}_{\{1\}^c}(|\psi\rangle\langle\psi|),\quad \rho_2 = \mathrm{tr}_{\{2\}^c}(|\psi\rangle\langle\psi|),\quad \ldots,\quad \rho_n = \mathrm{tr}_{\{n\}^n}(|\psi\rangle\langle\psi|)$$

sorted local spectra $\downarrow$

$$\underbrace{\left(\lambda_1^{(1)},\ldots,\lambda_{d_1}^{(1)},\quad \lambda_1^{(2)},\ldots,\lambda_{d_2}^{(2)},\quad \ldots,\quad \lambda_1^{(n)},\ldots,\lambda_{d_n}^{(n)}\right)}_{\in\mathbb{R}^{d_1+d_2+\ldots+d_n}}$$

Local Spectra: Qubits

Given a pure state of n qubits

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_n} x_{i_1, i_2, \ldots, i_n} |i_1\rangle|i_2\rangle \ldots |i_n\rangle \in \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$

One-particle reduced density matrices

$$\rho_1 = \mathrm{tr}_{\{1\}^c}(|\psi\rangle\langle\psi|), \quad \rho_2 = \mathrm{tr}_{\{2\}^c}(|\psi\rangle\langle\psi|), \quad \ldots, \quad \rho_n = \mathrm{tr}_{\{n\}^n}(|\psi\rangle\langle\psi|)$$

sorted local spectra $\downarrow$

$$\underbrace{\big(\lambda_1^{(1)}, \lambda_2^{(1)}, \qquad \lambda_1^{(2)}, \lambda_2^{(2)}, \qquad \ldots, \qquad \lambda_1^{(n)}, \lambda_2^{(n)}\big)}_{\in \mathbb{R}^{2n}}$$

Local Spectra: Qubits

Given a pure state of n qubits

$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_n} x_{i_1,i_2,\ldots,i_n} |i_1\rangle|i_2\rangle \ldots |i_n\rangle \in \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$

One-particle reduced density matrices

$$\rho_1 = \mathrm{tr}_{\{1\}^c}(|\psi\rangle\langle\psi|), \quad \rho_2 = \mathrm{tr}_{\{2\}^c}(|\psi\rangle\langle\psi|), \quad \ldots, \quad \rho_n = \mathrm{tr}_{\{n\}^n}(|\psi\rangle\langle\psi|)$$

sorted local spectra $\downarrow$

$$\underbrace{\big(\lambda_1^{(1)}, \quad \lambda_1^{(2)} \quad ,\ldots, \quad \lambda_1^{(n)}\big)}_{\in \mathbb{R}^n}$$

keep only the largest eigenvalue of each one-qubit reduced density matrix (or subtract the smallest eigenvalue from the largest)

SLOCC Equivalence

Two pure states of n particles

$$|\psi\rangle, |\phi\rangle \in \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n}$$

are SLOCC equivalent iff there is a sequence of local operations and classical communication (LOCC) that converts $|\psi\rangle$ with non-zero probability to $|\phi\rangle$ and vice versa.

$\Longleftrightarrow$ There exists $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, $p_1, p_2 > 0$ such that

$$p_1|\phi\rangle = (A_1 \otimes \ldots \otimes A_n)|\psi\rangle \quad \text{and} \quad p_2|\psi\rangle = (B_1 \otimes \ldots \otimes B_n)|\phi\rangle$$

W. l. o. g., there exist invertible $T_i \in \mathsf{SL}(d_i)$, $\mu \in \mathbb{C}$ such that

$$|\phi\rangle = \mu(T_1 \otimes \ldots \otimes T_n)|\psi\rangle$$

[W. Dür, G. Vidal, J. I. Cirac, PRA 62, 062314 (2000)]

SLOCC Invariants

If the pure states of n particles

$$|\psi\rangle, |\phi\rangle \in \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n}$$

are SLOCC equivalent, then there exists $\lambda \in \mathbb{C}$ such that for all invariants f of $\mathsf{SL}(d_1) \otimes \mathsf{SL}(d_2) \otimes \ldots \otimes \mathsf{SL}(d_n)$

$$f(\psi) = f(\lambda\phi).$$

- The algebra of polynomial invariants is generated by a finite (but huge) number of polynomials.
- Polynomial invariants do not suffice to decide SLOCC equivalence.
- For $n > 3$ qubits, there are infinitely many entanglement classes.

SLOCC Covariants

- polynomial invariants map a vector to a (homogeneous) polynomial function of the components
- covariants map a vector to a (homogeneous) polynomial function of the components times a representation of the group
- the representation of the group is associated with a highest weight vector
- covariants form a finitely generated algebra
- for n qubits, covariants can be encoded as polynomial in $2^n + 2n$ variables

$$f(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{C}[x_{i_1,\ldots,i_n}][y_0^{(1)}, y_1^{(1)}, \ldots, y_0^{(n)}, y_1^{(n)}]$$

- the weight $\boldsymbol{w}$ of a homogeneous covariant $f(\boldsymbol{x}, \boldsymbol{y})$ can be computed from the degrees in $\boldsymbol{x}$ and $\boldsymbol{y}$.

Entanglement Polytopes & Covariants

[M. Walter, B. Doran, D. Gross, M. Christandl, Science 340 (2013)]
[A. Sawicki, M. Oszmaniec, M. Kuś, RMP 26 (2014)]

- The local spectra of all states in the closure of an SLOCC orbit form a polytope, the *entanglement polytope*.
- What is more, the polytope is spanned by the normalized highest weight vectors of the covariants that do not vanish identically on an SLOCC orbit.
- It suffices to check the finitely many covariants that generate the algebra of all covariants.
- Hence, there are finitely many points in the ambient space of the polytopes that can be vertices.

$\Longrightarrow$ There are finitely many entanglement polytopes for any number of particles which provide a natural coarse-graining of the infinitely many entanglement classes.

Computing (Qubit) Covariants

[E. Briand, J.-G. Luque, J.-Y. Thibon, J. Phys. A 36 (2008)]

- the so-called ground form

$$f_0(x,y) = \sum_{i_1,\ldots,i_n} x_{i_1,\ldots,i_n} \cdot y_{i_1}^{(1)} \ldots y_{i_n}^{(n)}$$

  is an n-qubit covariant with normalized weight $(1,1,\ldots,1)$.

- all covariants can be computed from f_0 using so-called transvectants

- the algorithm terminates after a finite number of steps

n	#invariants	#covariants	#normalized weights
3	1	6	6
4	4	170	124
5	>124	>37886	>2574

Computing Entanglement Polytopes

main observation

a vertex $v \in \mathbb{R}^n$ is not contained in the entanglement polytope $\mathcal{P}(|\psi\rangle)$ of a state $|\psi\rangle$

$\Longleftrightarrow$ all covariants $f(\boldsymbol{x}, \boldsymbol{y})$ with normalized weight $\overline{\boldsymbol{w}}(f) = \boldsymbol{v}$ vanish identically

$\Longleftrightarrow$ all coefficients $c(\boldsymbol{x})$ of all covariants $f(\boldsymbol{x}, \boldsymbol{y})$ with $\overline{\boldsymbol{w}}(f) = \boldsymbol{v}$ vanish identically

$\Longleftrightarrow$ the state $|\psi\rangle$ lies in the variety $\mathrm{Var}(\mathcal{I}_{\boldsymbol{v}})$ of the ideal

$$\mathcal{I}_{\boldsymbol{v}} = \langle c(\boldsymbol{x})\colon c(\boldsymbol{x}) \in \mathsf{coeff}(f(\boldsymbol{x}, \boldsymbol{y})) \mid \overline{\boldsymbol{w}}(f) = \boldsymbol{v} \rangle$$

generated by the coefficients $c(\boldsymbol{x})$ of all $f(\boldsymbol{x}, \boldsymbol{y})$ with $\overline{\boldsymbol{w}}(f) = \boldsymbol{v}$

Computing Entanglement Polytopes

algorithm (basic idea)

1. start with the full entanglement polytope
2. remove one vertex $\boldsymbol{v}$ (up to symmetry)
3. compute the corresponding ideal $\mathcal{I}_{\boldsymbol{v}} \leq \mathbb{C}[\boldsymbol{x}]$, its radical $\sqrt{\mathcal{I}_{\boldsymbol{v}}}$, and its primary decomposition yielding the irreducible components of the variety $\mathrm{Var}(\mathcal{I}_{\boldsymbol{v}})$
4. test which covariants vanish on the irreducible components of the variety; this defines sub-polytopes, and the states in that entanglement polytope lie in the corresponding component of the variety
5. ensure that there is a state for which at least one covariant for each vertex is non-zero (compute ideal quotients)
6. continue in the same way with all sub-polytopes (up to symmetry)

Three Qubits

- six covariants with normalized weights

A_{111}	B_{200}	B_{020}	B_{002}	C_{111}	D_{000}
$(1,1,1)$	$(1,\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2},1)$	$(\frac{2}{3},\frac{2}{3},\frac{2}{3})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

- removing $v=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ yields a prime ideal with $D_{000}=0$
- removing $v=(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ yields an ideal with $C_{111}=0$ that has 3 associated primary ideals; for all $D_{000}=0$, and two of the covariants B_i vanish
- removing $v=(1,\frac{1}{2},\frac{1}{2})$ yields an ideal with $B_{200}=0$ that has 2 associated primary ideals; for all $C_{111}=D_{000}=0$, and one of the covariants B_{020} and B_{002} vanishes as well

$\Longrightarrow$ reproduces the known entanglement types of [Dür et al.]

Four Qubits

- 170 covariants with 124 normalized weights
- coefficients of the covariants are polynomials in $2^4 = 16$ variables, but $n = 4$ variables can be removed by local unitary transformations
- computation of the radical and the primary decomposition already become rather complicated

$\Longrightarrow$ like in [Walter et al. (2013)], successfully reproduced the known entanglement types of [Verstraete et al., PRA 65 (2002)]

Five Qubits

- we do not yet even know a generating set for all covariants; more than 37886 generators
- consider subset of all five-qubit states:

$$\mathcal{W}_{0,1,2} = \{\text{pure states with } 0,\ 1, \text{ or } 2 \text{ excitations}\}, \quad \dim \mathcal{W}_{0,1,2} = 16$$
$$\mathcal{W}_{0,2} = \{\text{pure states with } 0 \text{ or } 2 \text{ excitations}\}, \quad \dim \mathcal{W}_{0,2} = 11$$

- all invariants vanish on $\mathcal{W}_{0,2} \subset \mathcal{W}_{0,1,2}$, i.e., they provide no information
- complete set of 15733 non-vanishing covariants with 1903 different normalized weights on $\mathcal{W}_{0,1,2}$
- note that $\mathcal{W}_{0,1,2}$ is *not* invariant under SLOCC

Results for $\mathcal{W}_{0,2}$

12 different 5-dim. polytopes up to permutations, 128 in total

#vertices	#vertices of $\mathcal{P}_{\text{full}}$	#facets	$\#\mathrm{Aut}(\mathcal{P})$	#perms	$\dim \mathcal{I}$
26	26	16	120	1	11
27	25	17	6	20	8
27	24	17	12	10	5
23	23	16	12	10	8
23	23	19	12	10	8
21	21	18	4	30	7
20	20	22	8	15	6
20	20	18	12	10	6
17	17	17	12	10	6
16	16	26	120	1	7
14	14	20	12	10	5
11	11	11	120	1	5

Results for $\mathcal{W}_{0,2}$

- in addition to the vertices of the full polytope, the points $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4})$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4})$, together with permutations, may become vertices of entanglement polytopes
- those entanglement types extend to the variety of all states for which all other (not yet known) covariants vanish; in particular to $\mathcal{W}_{0,1,2}$
- the irreducible variety corresponding to an entanglement polytope may intersect $\mathcal{W}_{0,1,2}$ in multiple irreducible components
  consider, e.g., the pure product states in $\mathcal{W}_{0,1,2}$; are just the basis states
- so far, 390 candidate polytopes with dim. 5 for $\mathcal{W}_{0,1,2}$,
  but some might be the union of others, and some might be missing
  (currently processing 5712 sub-polytopes)

Summary & Outlook

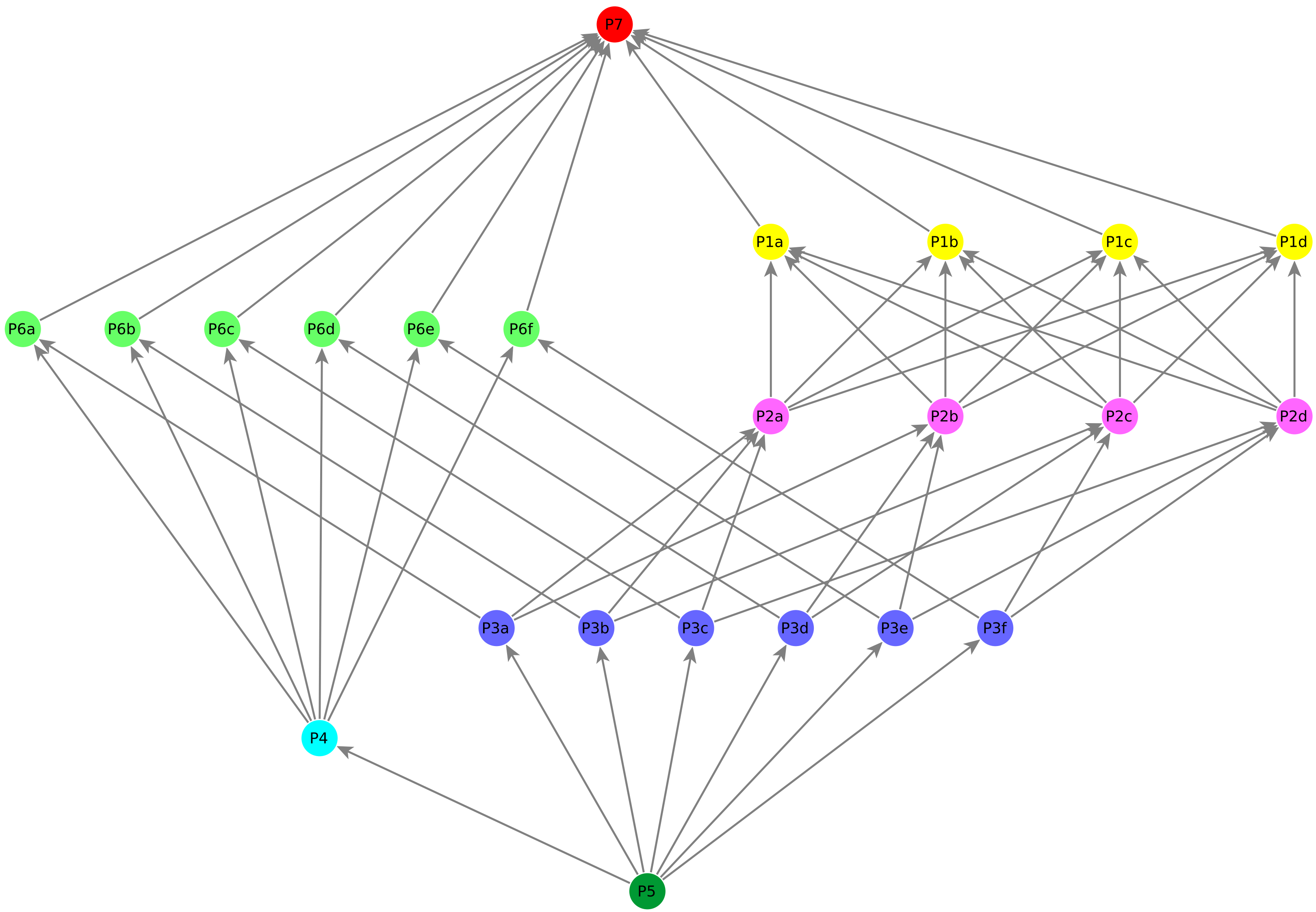
Summary

- coarse-grained SLOCC entanglement types from entanglement polytopes
- reproduction of known entanglement types for three and four qubits
- new partial results for five qubits

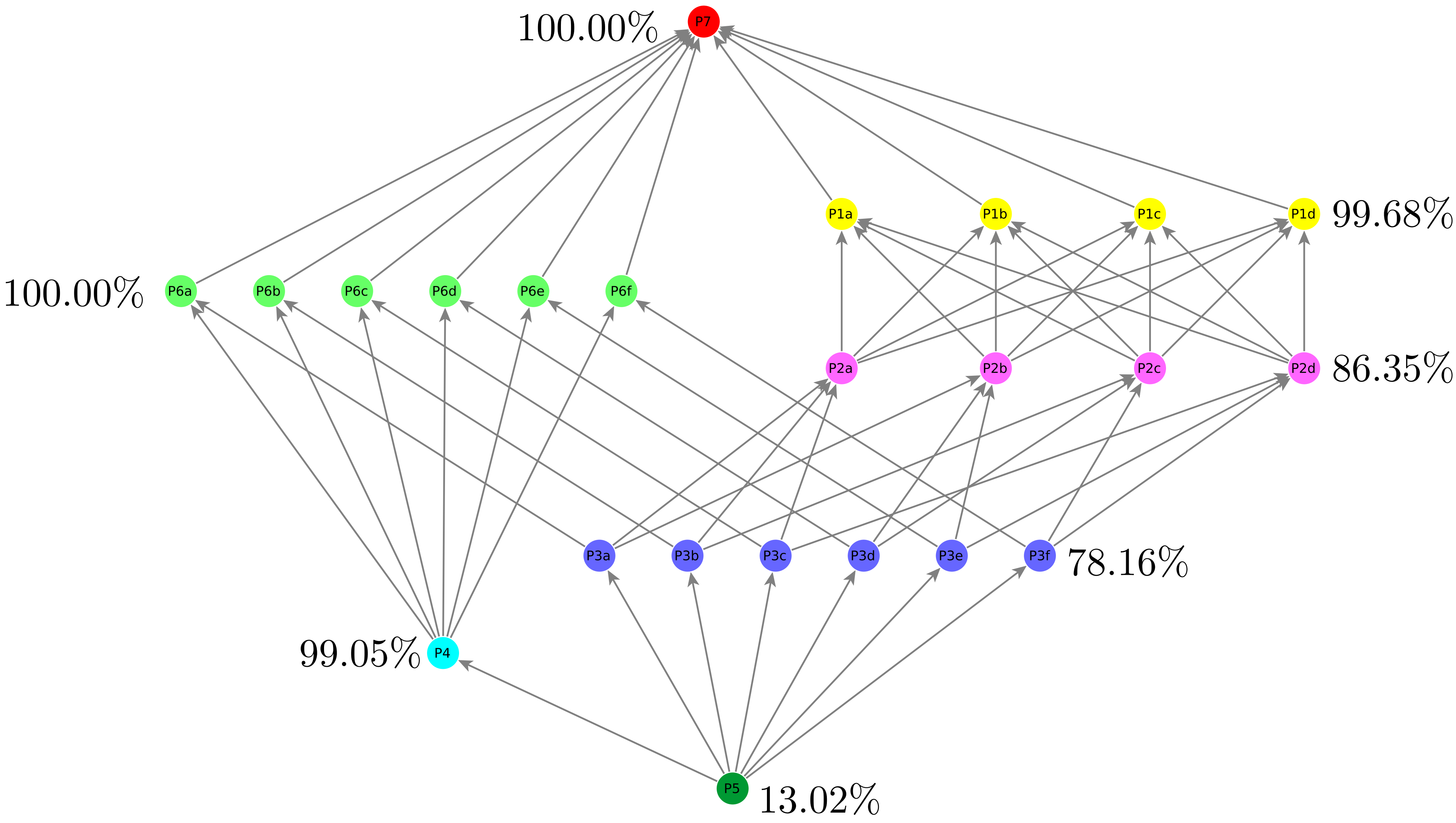
Outlook

- computation for $\mathcal{W}_{0,1,2}$ ongoing
- find more/complete set of covariants for five qubits
- derive similar results for three qutrits
  (structure of covariants is more complicated)
- analyse the lattice structure of the polytopes and their volume induced by the Haar measure on pure states (see also [arXiv:1502.05095])

Lattice of Four-Qubit Polytopes



Lattice of Four-Qubit Polytopes



Volumina of Four-Qubit Polytopes

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<tr><td>$\mathcal{P}_1$
996 761</td><td colspan="3">$\mathcal{P}_1^a$
990 140</td><td colspan="3">$\mathcal{P}_1^b$
990 137</td><td colspan="3">$\mathcal{P}_1^c$
990 204</td><td colspan="3">$\mathcal{P}_1^d$
990 262</td></tr>
<tr><td>$\mathcal{P}_2$
863 481</td><td colspan="3">$\mathcal{P}_2^a$
705 172</td><td colspan="3">$\mathcal{P}_2^b$
704 928</td><td colspan="3">$\mathcal{P}_2^c$
704 932</td><td colspan="3">$\mathcal{P}_2^d$
704 791</td></tr>
<tr><td>$\mathcal{P}_3$
781 562</td><td colspan="2">$\mathcal{P}_3^a$
607 121</td><td colspan="2">$\mathcal{P}_3^b$
607 010</td><td colspan="2">$\mathcal{P}_3^c$
607 176</td><td colspan="2">$\mathcal{P}_3^d$
606 791</td><td colspan="2">$\mathcal{P}_3^e$
606 925</td><td colspan="2">$\mathcal{P}_3^f$
607 051</td></tr>
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990 478</td><td colspan="12">$\mathcal{P}_4$
990 478</td></tr>
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130 165</td><td colspan="12">$\mathcal{P}_5$
130 165</td></tr>
<tr><td>$\mathcal{P}_6$
1 000 000</td><td colspan="2">$\mathcal{P}_6^a$
995 287</td><td colspan="2">$\mathcal{P}_6^b$
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