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Tomography of Quantum States in Small Dimensions

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Abstract

We consider the problem of determining the state of a finite dimensional quantum system by a finite set of different measurements in an optimal way. The measurements can either be projective von Neumann measurements or generalized measurements (POVMs). While optimal solutions for projective measurements are only known for prime power dimensions, based on numerical solutions it is conjectured that solutions for POVMs exist in any dimension. We support this conjecture by constructing explicit algebraic solutions in small dimensions d, in particular $d = 12$.

Keywords: Quantum state tomography, SIC-POVMs, MUBs, Weyl–Heisenberg group

1 Introduction

We consider a quantum system of finite dimension d. A state of that system can be described by a density matrix $\rho \in \mathbb{C}^{d \times d}$, which is a positive semi-

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definite Hermitian matrix with trace one. Hence, ρ can be described by d^2-1 independent real parameters. The problem of quantum state tomography is to determine these parameters by means of measurements.

Projective von Neumann measurements can be described by a set of mutually orthogonal projection operators Π_i whose sum is the identity matrix, i.e.

$$\Pi_i^2 = \Pi_i \qquad \text{for all } i, \tag{1}$$

$$\sum_i \Pi_i = I_d, \qquad \text{and} \tag{2}$$

$$\mathrm{Tr}(\Pi_i \Pi_j) = 0 \qquad \text{for } i \neq j. \tag{3}$$

The statistics of repeated measurements of the identically prepared state ρ yields expectation values

$$p_i := \Pr(\text{outcome} = i) = \mathrm{Tr}(\rho P_i). \tag{4}$$

If all the projection operators have rank one, the corresponding measurement has d different outcomes. Condition (2) implies that the sum of the probabilities (4) is one. Hence a projective measurement yields at most $d-1$ independent real parameters. Therefore at least $d+1$ different measurements are needed in order to obtain the d^2-1 independent real parameters of the density matrix ρ.

If we relax condition (3) and replace the orthogonal projection operators by arbitrary positive semi-definite operators E_j we arrive at the concept of so-called positive operator-valued measures (POVMs) (see, e.g., [11]). The conditions for a POVM read:

$$\sum_j E_j = I_d \qquad \text{and} \tag{5}$$

$$E_j \geq 0 \qquad \text{for all } j. \tag{6}$$

Quite often, the elements E_j will all have rank one and will be sub-normalized projection operators. Again, the statistics of the POVM yields non-negative values $\tilde{p}_j$

$$\tilde{p}_j := \Pr(\text{outcome} = j) = \mathrm{Tr}(\rho E_j). \tag{7}$$

While a projective measurement yields at most d parameters whose sum equals one, the number of elements E_j of a POVM is unbounded. Obviously, a minimal POVM that allows to completely determine the density matrix ρ must have at least d^2 elements.

2 Optimal Measurements

2.1 Projective Measurements

In order to achieve maximal independent measurement results, one requires that the outcomes of one projective measurement are independent of the other projective measurements. A projective measurement with the maximal number of d elements defines an orthonormal basis of the space $\mathbb{C}^d$. Two orthonormal bases $\mathcal{B}^k = \{|\psi_i^k\rangle : i = 1, \ldots, d\}$ and $\mathcal{B}^\ell = \{|\psi_j^\ell\rangle : j = 1, \ldots, d\}$ are called *mutually unbiased* iff

$$|\langle\psi_i^k|\psi_j^\ell\rangle|^2 = \begin{cases} 1/d & \text{for } k \neq \ell, \\ \delta_{i,j} & \text{for } k = \ell. \end{cases} \tag{8}$$

It has been shown that a collection of $d+1$ mutually unbiased bases provides an optimal means of determining the density matrix of a quantum system of dimension d [15].

Constructions for maximal sets of $d+1$ mutually unbiased bases (MUBs) are only known for prime power dimensions (see, e.g., [10] and references therein). Some constructions are related to finite affine planes [5]. The measurement outcomes can be interpreted as line integrals in the corresponding finite phase spaces [13,14].

For dimensions which are not a prime power, little is known about the maximal number of mutually unbiased based. For example, it is widely believed that there are no more than three MUBs in dimension six [6,16], while at least three MUBs exist in any dimension.

2.2 Generalized Measurements

As mentioned before, a generalized measurement that allows the reconstruction of the state ρ must have at least d^2 elements. A POVM with that property is called *informationally complete*. If in addition the measurement results are maximally independent, the POVM is called *symmetric informationally complete POVM* (SIC-POVM). It consists of d^2 operators of the form $E_j = \Pi_j/d$, where the rank-one projection operators $\Pi_j = |\phi_j\rangle\langle\phi_j|$ fulfill the conditions

$$\mathrm{Tr}(\Pi_j\Pi_k) = \frac{1}{d+1} \qquad \text{for } j \neq k. \tag{9}$$

It has been conjectured that SIC-POVMs exists for all dimensions [1,12,16]. Numerical solutions for dimension $d \leq 45$ are discussed in [12]. Zauner [16] provides algebraic solutions for $d = 2, 3, 4, 5$, and a solution for $d = 8$ based on the work of Hoggar [7]. Additional algebraic solutions for $d = 7, 19$ can be found in [1]. The first solution for a non prime-power dimension, $d = 6$, was given in [6]. Here we will present further algebraic solutions for small dimensions, including $d = 12$.

A common property of all solutions obtained so far is that they are highly symmetric. The elements of the SIC-POVM (with the exception of the solution for $d = 8$) are a single orbit of the finite version of the Weyl–Heisenberg group generated by a cyclic shift operator and a shift in the Fourier transformed basis. All of these SIC-POVMs possess at least an additional symmetry of order three.

In the following we will construct SIC-POVMs which have a prescribed symmetry group. In addition to SIC-POVMs which are constructed using the Weyl–Heisenberg group, other symmetry groups which correspond to so-called nice unitary error bases [9] are used as well.

3 Construction of SIC-POVMs with Symmetry

In order to simplify the construction of SIC-POVMs, we will restrict our attention to SIC-POVMs that are group-covariant with respect to a finite symmetry group [12]. Given a finite group G which we will identify with a unitary representation of degree d, a POVM $\mathcal{P}$ in dimension d is covariant with respect to G iff

$$\mathcal{P} = \{U_g^{-1} E_0 U_g : g \in G\} \quad \text{for some } E_0 \in \mathcal{P}. \tag{10}$$

If the representation of the group is irreducible, the orbit of any non-zero operator E_0 under G is up to normalization a POVM, as by the lemma of Schur the sum $\sum_{g\in G} U_g^{-1} E_0 U_g$ is proportional to the identity matrix. As a SIC-POVM has d^2 elements, the order of the group G must be a multiple of d^2.

One candidate for the group G is the so-called Weyl–Heisenberg group H_d which exists for any dimension d. It is generated by the following two operators:

$$X := \sum_{j=0}^{d-1} |j+1\rangle\langle j| \qquad \text{and} \qquad Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|,$$

where $\omega_d := \exp(2\pi i/d)$ is a primitive complex d-th root of unity and the cyclic shift is modulo d. Each element of H_d can be uniquely written as $\omega_d^c X^a Z^b$ with $a, b, c \in \{0, \ldots, d-1\}$. Two elements $\omega_d^c X^a Z^b$ and $\omega_d^{c'} X^{a'} Z^{b'}$ commute iff $ab' - a'b = 0 \bmod d$. The center $\zeta(H_d)$ of the group H_d is generated by $\omega_d I$, where I denotes the identity matrix. Ignoring the global phase factor, the group H_d is isomorphic to the direct product of two cyclic groups of order d, i.e., $H_d/\zeta(H_d) \cong \mathbb{Z}_d \times \mathbb{Z}_d$ where $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$ denotes the ring of integers modulo d. The matrices $X^a Z^b$ are mutually orthogonal with respect to the trace inner product and form a vector space basis of all $d \times d$ matrices. Later we will also use other groups G with the property that the quotient $\widetilde{G}$ of G by its center has d^2 elements and that the corresponding matrices form a vector space basis of all $d \times d$ matrices. Such groups are known as *abstract error groups*, and the matrices corresponding to $\widetilde{G}$ are called *nice unitary error bases* (see [9]). The group $\widetilde{G}$ is called *index group* as the elements of the POVM (10) can be labeled by the elements of $\widetilde{G}$.

It has been conjectured that SIC-POVMs which are group-covariant with respect to the Weyl–Heisenberg group exist in any dimension [12]. Zauner [16] has already conjectured that one can always find a SIC-POVM that possesses an additional symmetry of order three that stabilizes the initial operator E_0. This additional symmetry is a particular element of the normalizer of H_d in the full unitary group, which is known as the Jacobi group J_d [6] or Clifford group [1]. The action of J_d on H_d modulo the center via conjugation is isomorphic to $\mathrm{SL}(2, \mathbb{Z}_d)$, the group of 2×2 matrices over the integers modulo d with unit determinant. Appleby [1] has verified that indeed all numerical solutions computed by [12] have a symmetry that is conjugated to the element given by Zauner [16]. Based on those results, Appleby conjectured that any SIC-POVM that is covariant with respect to the Weyl–Heisenberg group would possess an additional symmetry that is conjugated to Zauner's element. However, in Section 4.1 we will provide a counter-example to this strongest version of Appleby's conjectures.

We use the following ansatz:

Algorithm 1 (Searching for a group-covariant SIC-POVM)

(i) *Let $G \subset U(d)$ be a unitary representation of degree d of an abstract error group.*

(ii) *Let T be a non-trivial element of the normalizer of G in the full unitary group, i.e. $T \notin G$ and T is not proportional to identity.*

(iii) *Let $\mathcal{B} = \{|b_0\rangle, \ldots, |b_{m-1}\rangle\}$ be a basis of an eigenspace of T.*

(iv) *A generic vector in this eigenspace has the form*

$$|\phi_0\rangle = \sum_{j=0}^{m-1} (x_{2j} + i x_{2j+1}) |b_j\rangle, \tag{11}$$

where $x_0, \ldots, x_{2m-1}$ are real variables and $i^2 = -1$.

(v) *Let* $\mathcal{T} = \{g_0 = id, \ldots, g_{d^2-1}\}$ *be representatives of the cosets of* G *by its center* $\zeta(G)$.

(vi) *Combining (9), (10), and (11), we get a system of polynomial equations for the variables* x_μ, *in particular*

$$\text{(12)} \qquad \operatorname{Tr}(|\phi_0\rangle\langle\phi_0|U_{g_j}^{-1}|\phi_0\rangle\langle\phi_0|U_{g_j}) = \frac{1}{d+1} \quad \textit{for all } g_j \in \mathcal{T} \setminus \{id\}.$$

(vii) *Try to find a solution for the real variables* x_μ.

Note that we have to double the number of variables as complex conjugation cannot directly be expressed as a polynomial function. However, we can set e.g. $x_1 = 0$ as we may multiply $|\phi_0\rangle$ by any phase $e^{i\theta}$. The restriction of the fiducial vector $|\phi_0\rangle$ to an eigenspace of T reduces the number of variables. This is important as it turns out that the system of polynomial equations is quite hard to solve. Another technical problem is that the variety of the solutions is given over the algebraically closed field $\mathbb{C}$, but we are only interested in the solutions over $\mathbb{R}$. Nonetheless, using the computer algebra system MAGMA [2], we were able to compute some new SIC-POVMs. Further details will be given in the next section.

4 Examples

4.1 Weyl–Heisenberg Group

In addition to the previously known algebraic solutions for dimension $d = 2, 3, 4, 5, 7, 13$ [1,16], we have computed SIC-POVMs that are covariant with respect to the Weyl–Heisenberg group for $d = 6, 8, 9, 10, 12, 13$ (see Table 1). Unfortunately, most of the solutions are quite complicated, and further investigation is necessary. All solutions can be obtained from the author on request.

In all cases, the fiducial vector is stabilized by an element of order 3 or 6. The last column lists the number of SIC-POVMs obtained from the action of the Jacobi group. With the exception of $d = 8$, $d = 12$, and possibly $d = 13$, complex conjugation doubles the number of SIC-POVMs.

In the following, we focus on dimension $d = 12$. One solution for the not normalized fiducial vector $|\phi_0\rangle = |\psi_{12}\rangle$ is given in Table 2. The coordinates v_i are elements of the number field $\mathbb{Q}(\sqrt{2}, \sqrt{13}, \theta_1, \theta_2, i, \omega_3)$ of degree 64 generated by

$$\theta_1 := \sqrt{\sqrt{13} - 1}, \quad \theta_2 := \sqrt{\sqrt{13} + 3}, \quad i^2 = -1, \quad \omega_3 := \exp 2\pi i/3.$$

Table 1
SIC-POVMs which are covariant with respect to the Weyl–Heisenberg group.

d	unitary automorphism	number of SIC-POVMs
6	order 3	$48 + 48$
8	order 6	64
9	order 3	$216 + 216$
10	order 3	$240 + 240$
12	order 3	384
13	order ≥ 3	not yet computed

The initial vector $|\psi_{12}\rangle$ is an eigenvector of the matrix T_{12} given in Table 3. This matrix T_{12} is not conjugated to Zauner's matrix, as the multiplicities of the eigenvalues are 3, 3, and 6, whereas Zauner's matrix has multiplicities 3, 4, and 5 (see [16]). Hence we have a counter-example to Conjecture C of [1] which states that any SIC-POVM that is covariant with respect to the Weyl–Heisenberg group possesses a symmetry of order three that is conjugated to Zauner's matrix.

Before considering other symmetry groups in the next section, we note that for dimension $d = 13$ the additional symmetry T_{13} can be chosen to be a permutation matrix. This helped in solving the system of polynomial equations, as the eigenvectors of T_{13} have a particular simple structure.

4.2 Other Groups

As mentioned before, the orbit of a vector under a finite irreducible matrix group gives rise to a covariant POVM. Candidates for symmetry groups are abstract error groups. A catalogue of the corresponding representations for small dimension can be found at [8].

An algebraic solution for a SIC-POVM in dimension 8 has been constructed by Hoggar [7, Example 8]. An explicit expression can be found in [16]. That SIC-POVM is covariant with respect to the threefold tensor product of the group generated by the Pauli matrices. The corresponding index group is the elementary abelian group of order 64.

Numerical solutions for SIC-POVMs which are covariant with respect to other groups have been reported in [12]. Using the numbering of small groups as e.g. in MAGMA, the corresponding index groups are $G(36, 11)$, $G(64, 8)$, and $G(81, 9)$ where we use the notation $G(n, m)$ for `SmallGroup(n,m)`. Note that in [12] additionally the group $G(36, 14)$ is listed, but this group corresponds to the Weyl–Heisenberg group.

Table 2
Coordinates of the (not normalized) initial vector $|\psi_{12}\rangle = \sum_{i=1}^{12} v_i|i\rangle$.

$$
\begin{aligned}
v_1 &= 16\\
v_2 &= \Big(((\sqrt{26}+\sqrt{2}-\sqrt{13}-1)\theta_1+(\sqrt{26}-5\sqrt{2}-2\sqrt{13}+10))\theta_2\\
&\qquad +((-\sqrt{26}-3\sqrt{2}+2\sqrt{13}+6)\theta_1+(4\sqrt{2}-4))\Big)i\\
&\quad +((-\sqrt{13}-1)\theta_1+(\sqrt{26}-5\sqrt{2}))\theta_2+(\sqrt{26}+3\sqrt{2})\theta_1+4\\
v_3 &= ((4\sqrt{2}-8)\theta_1-4\sqrt{26}-4\sqrt{2}+4\sqrt{13}+4)i\\
v_4 &= (((4\sqrt{2}-4)\theta_1-4\sqrt{2}+8)\theta_2+(8\sqrt{2}-8))i+(-4\theta_1-4\sqrt{2})\theta_2+8\\
v_5 &= (-2\sqrt{26}-6\sqrt{2})\theta_1-8\\
v_6 &= \Big(((\sqrt{26}-\sqrt{2}-\sqrt{13}+1)\theta_1+(2\sqrt{2}-4))\theta_2\\
&\qquad +((-2\sqrt{2}+4)\theta_1+(2\sqrt{26}+2\sqrt{2}-2\sqrt{13}-2))\Big)i\\
&\quad +((-\sqrt{13}+1)\theta_1+2\sqrt{2})\theta_2+2\sqrt{2}\theta_1+2\sqrt{13}+2\\
v_7 &= (16\sqrt{2}-16)i\\
v_8 &= \Big(((\sqrt{26}+\sqrt{2}-\sqrt{13}-1)\theta_1+(\sqrt{26}-5\sqrt{2}-2\sqrt{13}+10))\theta_2\\
&\qquad +((\sqrt{26}+3\sqrt{2}-2\sqrt{13}-6)\theta_1-4\sqrt{2}+4)\Big)i\\
&\quad +((-\sqrt{13}-1)\theta_1+(\sqrt{26}-5\sqrt{2}))\theta_2+(-\sqrt{26}-3\sqrt{2})\theta_1-4\\
v_9 &= -4\sqrt{2}\theta_1-4\sqrt{13}-4\\
v_{10} &= (((4\sqrt{2}-4)\theta_1-4\sqrt{2}+8)\theta_2+(-8\sqrt{2}+8))i+(-4\theta_1-4\sqrt{2})\theta_2-8\\
v_{11} &= ((2\sqrt{26}+6\sqrt{2}-4\sqrt{13}-12)\theta_1-8\sqrt{2}+8)i\\
v_{12} &= \Big(((\sqrt{26}-\sqrt{2}-\sqrt{13}+1)\theta_1+(2\sqrt{2}-4))\theta_2\\
&\qquad +((2\sqrt{2}-4)\theta_1-2\sqrt{26}-2\sqrt{2}+2\sqrt{13}+2)\Big)i\\
&\quad +((-\sqrt{13}+1)\theta_1+2\sqrt{2})\theta_2-2\sqrt{2}\theta_1-2\sqrt{13}-2
\end{aligned}
$$

Below we give algebraic solutions for the non-abelian groups $G(36,11)$ and $G(64,78)$. The SIC-POVM for the latter group appears to be new.

Table 3
The matrix T_{12} stabilizing $|\psi_{12}\rangle$ is given as 1/2 times the following matrix, where $\omega_{12} = \exp(2\pi i/24)$ denotes a primitive complex 24th root of unity:

$$\left(\begin{array}{cccccccccccc}
\omega_{24}^{17} & 0 & 0 & -\omega_{24}^{7}+\omega_{24}^{3} & 0 & 0 & \omega_{24}^{5} & 0 & 0 & \omega_{24}^{11} & 0 & 0 \\
0 & \omega_{24}^{8} & 0 & 0 & -\omega_{24}^{4}+1 & 0 & 0 & \omega_{24}^{8} & 0 & 0 & -\omega_{24}^{4}+1 & 0 \\
0 & 0 & \omega_{24}^{5} & 0 & 0 & -\omega_{24}^{7}+\omega_{24}^{3} & 0 & 0 & \omega_{24}^{17} & 0 & 0 & \omega_{24}^{11} \\
\omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 \\
0 & \omega_{24}^{11} & 0 & 0 & \omega_{24}^{17} & 0 & 0 & -\omega_{24}^{7}+\omega_{24}^{3} & 0 & 0 & \omega_{24}^{5} & 0 \\
0 & 0 & -\omega_{24}^{4}+1 & 0 & 0 & \omega_{24}^{8} & 0 & 0 & -\omega_{24}^{4}+1 & 0 & 0 & \omega_{24}^{8} \\
\omega_{24}^{17} & 0 & 0 & \omega_{24}^{11} & 0 & 0 & \omega_{24}^{5} & 0 & 0 & -\omega_{24}^{7}+\omega_{24}^{3} & 0 & 0 \\
0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 \\
0 & 0 & \omega_{24}^{5} & 0 & 0 & \omega_{24}^{11} & 0 & 0 & \omega_{24}^{17} & 0 & 0 & -\omega_{24}^{7}+\omega_{24}^{3} \\
-\omega_{24}^{4}+1 & 0 & 0 & \omega_{24}^{8} & 0 & 0 & -\omega_{24}^{4}+1 & 0 & 0 & \omega_{24}^{8} & 0 & 0 \\
0 & -\omega_{24}^{7}+\omega_{24}^{3} & 0 & 0 & \omega_{24}^{17} & 0 & 0 & \omega_{24}^{11} & 0 & 0 & \omega_{24}^{5} & 0 \\
0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8} & 0 & 0 & \omega_{24}^{8}
\end{array}\right)$$

4.2.1 Dimension 6

We start with the group G_6 defined as

$$G_6 := \left\langle \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0 & -\omega_{12}^3 & 0 & 0 & 0 & 0\\ -\omega_{12}^3 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\omega_{12}^2 & 0 & 0\\ 0 & 0 & \omega_{12}^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \omega_{12} & 0\\ 0 & 0 & 0 & 0 & 0 & -\omega_{12} \end{pmatrix} \right\rangle,$$

where $\omega_{12} := \exp(2\pi i/12)$ denotes a primitive complex 12th root of unity. The irreducible matrix group G_6 has order 216 and is a representation of $G(216, 42)$. The index group $\widetilde{G}_6 := G_6/\zeta(G_6)$ is isomorphic to $G(36, 11)$. The following two elements normalize the group G_6:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -\omega_{72}^{17} & \omega_{72}^{71} & 0 & 0 & 0 & 0\\ -\omega_{72}^{17} & -\omega_{72}^{71} & 0 & 0 & 0 & 0\\ 0 & 0 & \omega_{72}^{5} & \omega_{72}^{23} & 0 & 0\\ 0 & 0 & \omega_{72}^{5} & -\omega_{72}^{23} & 0 & 0\\ 0 & 0 & 0 & 0 & \omega_{72}^{5} & \omega_{72}^{23}\\ 0 & 0 & 0 & 0 & \omega_{72}^{5} & -\omega_{72}^{23} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\omega_{24}^{3} & 0 & 0\\ 0 & 0 & -\omega_{24}^{9} & 0 & 0 & 0\\ 0 & -\omega_{24}^{11} & 0 & 0 & 0 & 0\\ \omega_{24}^{5} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \omega_{24}^{7}\\ 0 & 0 & 0 & 0 & -\omega_{24} & 0 \end{pmatrix}.$$

The group N_6 generated by G_6 and the previous matrices has order 3888, and the index group $\widetilde{G}_6 = N_6/G_6$ is isomorphic to $G(18, 3)$. As prescribed additional symmetry we use the matrix T_6 given in Table 4. A scalar multiple of T_6 is contained in N_6,

Solutions for a fiducial vector in an eigenspace of T_6 can be expressed over the number field $\mathbb{Q}(\omega_{72}, \theta)$ of degree 288 where $\omega_{72} := \exp 2\pi i/72$, $\theta := \sqrt[3]{\gamma}$,

Table 4
The matrix T_6 stabilizing $|\psi_6\rangle\langle\psi_6|$.

$$T_6 := \frac{1}{2}\begin{pmatrix} i-1 & -i-1 & 0 & 0 & 0 & 0 \\ 1-i & -i-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i-1 & i+1 & 0 & 0 \\ 0 & 0 & i-1 & -i-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_{12}^3+\omega_{12}^2-\omega_{12} & -\omega_{12}^3+\omega_{12}^2+\omega_{12} \\ 0 & 0 & 0 & 0 & -\omega_{12}^3-\omega_{12}^2+\omega_{12} & -\omega_{12}^3+\omega_{12}^2+\omega_{12} \end{pmatrix}$$

and γ is a root of the polynomial

$$\gamma^4 + \frac{-5\omega_{12}^3+4\omega_{12}}{252}\gamma^3 + \frac{3\omega_{12}^2-7}{49392}\gamma^2 - \frac{4\omega_{12}^3+\omega_{12}}{18670176}\gamma - \frac{\omega_{12}^2}{5489031744}$$

The coordinates of one of the four fiducial vectors which he have found are given in Table 5. Each of the vectors is only stabilized by T_6 which has order three, and the orbits of the vectors under N_6 are disjoint. Hence we get $18/3 = 6$ different SIC-POVMs from each of the four fiducial vectors. None of the SIC-POVMs is invariant under complex conjugation, so in total we obtain 48 SIC-POVMs which are covariant with respect to G_6.

4.2.2 Dimension 8

We have computed an algebraic solution for a SIC-POVM in dimension 8 that is covariant with respect to the Weyl–Heisenberg group (see Table 1). Here we consider the group G_8 of order 128 which is isomorphic to $G(128, 813)$ defined as

$$G_8 := \left\langle \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \end{pmatrix} \right\rangle,$$

where $i = \sqrt{-1}$. The center $\zeta(G_8)$ of G_8 has order two and is generated by $-I$. Hence the quotient $\widetilde{G}_8 = G_8/\zeta(G_8)$ is a group of order 64 isomorphic to $G(64, 78)$ which is a non-abelian index group. The normalizer of G_8 is the

Table 5
Coordinates of one of the (not normalized) initial vectors $|\psi_6\rangle$.

$$
\begin{aligned}
v_1 &:= 1\\
v_2 &:= \omega_6 - \omega_{12}\\
v_3 &:= (6223392\omega_9^2 - 2074464\omega_{18})\theta^9 + (16464\omega_{36}^{11} + 90552\omega_{36}^5)\theta^6\\
&\quad +(-658\omega_9^2 - 140\omega_{18})\theta^3 + (-\omega_{36}^{11} + 2\omega_{36}^5)/9\\
v_4 &:= (6223392\omega_{36}^{11} - 4148928\omega_9^2 - 2074464\omega_{36}^5 + 6223392\omega_{18})\theta^9\\
&\quad +(-107016\omega_{36}^{11} + 107016\omega_9^2 + 16464\omega_{36}^5 - 16464\omega_{18})\theta^6\\
&\quad +(-658\omega_{36}^{11} + 798\omega_9^2 - 140\omega_{36}^5 - 658\omega_{18})\theta^3 + (-\omega_{36}^{11} + \omega_9^2 - \omega_{36}^5 + \omega_{18})/9\\
v_5 &:= (6316742880i - 4094991936\omega_6 + 4094991936\omega_{12} + 10411734816)\theta^{11}/19\\
&\quad +(-192752280i + 83670048\omega_6 + 83670048\omega_{12} + 109082232)\theta^8/19\\
&\quad +(-326928i + 931686\omega_6 - 931686\omega_{12} - 1258614)\theta^5/19\\
&\quad +(-1477i - 4543\omega_6 - 4543\omega_{12} + 6020)\theta/57^2\\
v_6 &:= (6316742880i - 16728477696\omega_6 + 4094991936\omega_{12} + 2221750944)\theta^{11}/19\\
&\quad +(25412184i + 83670048\omega_6 - 301834512\omega_{12} + 109082232)\theta^8/19\\
&\quad +(-326928i + 1585542\omega_6 - 931686\omega_{12} + 604758)\theta^5/19\\
&\quad +(10563i - 4543\omega_6 - 7497\omega_{12} + 6020)\theta^2/57
\end{aligned}
$$

where $i^2 = -1$ and $\omega_m := \exp(2\pi i/m)$ denotes a primitive complex m-th root of unity.

group N_8 of order 4096 given by

$$
N_8 := \left\langle G_8, \begin{pmatrix} 0&0&0&0&0&0&0&-i\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&i&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} i&0&0&0&0&0&0&0\\ 0&i&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&0&0&i \end{pmatrix},\right.
$$
$$
\left. \begin{pmatrix} 0&0&0&0&0&0&-i&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-i&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0\\ i&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&i&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&i&0&0&0&0&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&i&0&0 \end{pmatrix} \right\rangle.
$$

The action of N_8 on G_8 via conjugation gives rise to the full automorphism group of G_8 of order 1024. In this case, we have not found a solution with an additional non-trivial symmetry. However, we can directly solve the polynomial equations (12) for a generic vector in $\mathbb{C}^8$ setting some coordinates to zero. One solution is

$$|\psi_8\rangle := \frac{1}{\sqrt{6}}(0, 1, \omega_8, 0, 1, \sqrt{2}, 0, \omega_8)^t, \tag{13}$$

where $\omega_8 := \exp(2\pi i/8)$ is a primitive complex 8th root of unity. Only the elements in the center of N_8 stabilize the fiducial vector given by (13). So the orbit of $|\psi_8\rangle\langle\psi_8|$ under the group N_8 has 1024 elements, which can be partitioned into 16 SIC-POVMs. On those 16 SIC-POVMs, the group N_8 acts as elementary abelian group $\mathbb{Z}_2^4$, corresponding to the outer automorphisms of G_8. Finally, we note that the set of 16 SIC-POVMs is invariant under complex conjugation, i.e., the SIC-POVM possesses a non-trivial anti-unitary automorphism of order two.

5 Conclusion & Outlook

The problem of completely identifying a quantum state by means of measurements can be tackled using either projective von Neumann measurements or generalized measurements (POVMs). Optimal projective measurements are closely connected to maximal sets of mutually unbiased bases (MUBs), and for POVMs we get the notion of SIC-POVMs. The situation for the two cases is quite different. Constructions for maximal sets of MUBs are known for any prime power dimension. So far, we do not know a general construction for SIC-POVMs for a sequence of arbitrary large dimensions. Imposing additional symmetries, we were able to explicitly construct SIC-POVMs in small dimensions, using various abstract error groups. Despite our initial hope, we were not able to derive a general construction of SIC-POVMs which are covariant with respect to the Weyl–Heisenberg group from the explicit algebraic solutions in small dimensions. Yet, our algebraic solutions are completely in line with the weakest formulation of Zauner's conjecture, namely that such SIC-POVMs exists in any dimension.

We conclude by mentioning a modification of the problem of determining the state of a quantum system. The modified problem is related to what is known in the literature as *Pauli problem*. Pauli posed the question whether knowledge of the probability distribution for both position and momentum would completely determine the wave function of a quantum system. Translated into our context, the question is whether the statistics of two projective measurements that correspond to two mutually unbiased bases suffices to determine a density operator that has rank one. That is, the reconstruction of the quantum state from the measurement statistics can make use of the additional promise that the density matrix has rank one.

In both cases, the answer is negative, i.e., we need more than two projective measurements in order to determine the $2d - 2$ independent real parameters of a pure quantum state. This follows directly from the existence of at least three MUBs in any dimension. All states from the third basis will have the same flat distribution with respect to the other two bases. For POVMs, it is also still open how many elements a minimal POVM must have to determine the $2d - 2$ independent real parameters of a pure quantum state [3,4].

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