Stabilization of quantum states in quantum-optical systems

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Recently it has been shown by several authors that quantum states can be stabilized against decoherence by periodically performing measurements and unitary transformations. We extend their decay model to a more realistic one in the framework of quantum optics conforming to the Markov approximation. It is shown that all quantum error correcting codes are able to cope with this decay model too. [S1050-2947(96)00410-6]

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I. INTRODUCTION

Much of the ongoing research is carried out to create and manipulate n-particle entangled states (ES). ES have applications in a variety of contexts: They are fundamental for quantum computation and quantum communication [1], are the basis of fundamental tests of quantum mechanics versus local realists' theories [2], and promise novel atomic spectroscopy techniques with resolution better than the standard quantum limit [3]. The main obstacle, however, is the fragility of macroscopic quantum states due to the coupling to an environment. Thus, it is of crucial importance to understand the effects of decoherence and to develop methods to stabilize ES against decoherence in realistic systems [4].

The generic device for creating and manipulating entanglement is the quantum computer, which may be viewed as a quantum state synthesizer. It is thus natural to discuss the problem of stabilizing ES in the language of quantum computing. The memory of a quantum computer (''quantum register'') consists of a set of two-level systems usually referred to as ''qubits.'' In order to perform any desired computation one should be able to perform arbitrary unitary transformations on the quantum register. These operations can be decomposed into a sequence of steps involving only a few qubits at a time (''quantum gates'') [1]. Further requirements to realizing a quantum computer are a way to read out the final result and a method to reset the quantum register.

There are two kinds of errors that arise in the process of creating and preserving ES: (i) Static errors that take place after the ES has been created successfully (''quantum memory errors'') and (ii) dynamical errors in the process of creating an ES (''quantum gate errors''). Schemes to deal with errors of type (i) have been proposed by several authors [5–9]. Very recently a scheme to correct dynamical errors of type (ii) has been proposed by Cirac *et al.* [10].

The error correction schemes proposed in Refs. [5–7] have the capability to correct single errors and require systems of $n=9$, 7, or 5 qubits to encode a single qubit. For these schemes, it has been shown that errors can be corrected perfectly if a *single* (but arbitrary) qubit interacts with the environment according to an arbitrary time evolution operator U_{dec}:

$$U_{\mathrm{dec}}:\begin{bmatrix}|0\rangle|e_0\rangle\\ |1\rangle|e_0\rangle\end{bmatrix}\rightarrow\begin{bmatrix}|0\rangle|e_1\rangle+|1\rangle|e_2\rangle\\ |0\rangle|e_3\rangle+|1\rangle|e_4\rangle\end{bmatrix}. \tag{1}$$

Here $|0\rangle$ and $|1\rangle$ are the basis states of the qubit and $|e_i\rangle$ are arbitrary states of the environment, as long as unitarity in Eq. (1) is fulfilled. The main idea of these error correction schemes is to measure periodically if, and what kind of, an error has taken place without gaining any information on the stored quantum superposition. If an error is detected an appropriate unitary transformation is performed to reconstruct the original state.

An example for such a code is that given by Shor [5], where a single qubit is encoded redundantly in nine qubits:

$$\underline{|0\rangle}=|+\rangle|+\rangle|+\rangle, \tag{2}$$

$$\underline{|1\rangle}=|-\rangle|-\rangle|-\rangle,$$

where

$$|\pm\rangle=\frac{1}{\sqrt{2}}(|0\rangle|0\rangle|0\rangle\pm|1\rangle|1\rangle|1\rangle).$$

In reality, of course, *all* qubits undergo decoherence at the same time. However, to first order in the interaction time t the total time evolution operator acting on all n qubits can be written as

$$U_{\mathrm{dec}}^{\mathrm{total}}(t)=\mathbb{1}-i(H_{\mathrm{dec}}^{(1)}+\cdots+H_{\mathrm{dec}}^{(n)})t+o(t^2), \tag{3}$$

where $H_{\mathrm{dec}}^{(i)}$ is the time invariant Hamiltonian describing the interaction between the ith qubit and its environment. In writing Eq. (3) we have assumed that each qubit interacts with independent environments. Up to first order in t, $U_{\mathrm{dec}}^{\mathrm{total}}(t)$ is a sum of operators that act only on a single qubit and its environment. Thus, by the linearity of quantum mechanics, errors changing all qubits at the same time can be corrected perfectly up to first order in t. It can be shown easily that if N error corrections are performed within a time interval $[0,T)$ there is a remaining error probability of order $o((T/N)^2)$ after each error correction event. Thus the accu-

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mulated error at time T is of order $o(N(T/N)^2)$. This error can be made arbitrarily small by choosing a sufficiently large N.

In most quantum-optical systems of practical interest, however, the system gets entangled very rapidly with the environment compared to typical experimental time scales. The time evolution of the interaction of the quantum bits with the environment is in an exponential regime. This is the limit where a Markov or white noise approximation can be made [11]. The resulting time evolution cannot be derived from a coherent, Hamiltonian evolution. Spontaneous emission from an excited atom is a typical example for a system where the Markov approximation is excellent. For a system of n qubits, each coupled to an individual environment, the unitary time evolution operator describing the interaction of the qubits with the environment cannot be decomposed to first order in the time t into a sum of operators, each acting only on a single qubit and its corresponding environment as in Eq. (3). Thus a frequent repetition of the error correction procedure may not freeze the state of the system in the same way as for errors deriving from a coherent, Hamiltonian evolution. The purpose of this paper is to show that all quantum error correction schemes, e.g., that presented in Refs. [5–7], remain valid for decoherence of this kind.

The paper is organized as follows. In Sec. II the master equation and quantum trajectory techniques to deal with quantum optical systems are briefly reviewed in order to establish the notation and methodology of the present paper. Error correction in this decay model is discussed asymptotically in Sec. III. Section IV deals with the correction of multiple errors.

II. DECAY AND DECOHERENCE ARE QUANTUM OPTICAL SYSTEMS

A. Master equation

In quantum-optical systems decay and decoherence are usually described within the theory of quantum Markov processes [11]. The physical basis of this theory is the assumption of a weak coupling of the system to an ''environment'' with a short correlation time. In the system dynamics this coupling will introduce damping and fluctuations. Under these assumptions the environment can be eliminated from the problem and a master equation of Lindblad form for the reduced density operator $\rho(t)$ can be derived ($\hbar=1$) [11]:

$$\frac{d}{dt}\rho(t)=-i[H_{\rm eff}\rho(t)-\rho(t)H_{\rm eff}^{\dagger}]+\sum_{j=1}^{k}\mathcal{J}_j\rho(t)\mathcal{J}_j^{\dagger}. \quad (4)$$

The reduced density operator $\rho(t)$ is obtained by tracing over the degrees of freedom of the environment. $H_{\rm eff}$ is a non-Hermitian, effective Wigner-Weisskopf Hamiltonian defined by

$$H_{\rm eff}=H_{\rm sys}-\frac{i}{2}\sum_{j=1}^{k}\mathcal{J}_j^{\dagger}\mathcal{J}_j,$$

where $H_{\rm sys}$ is the Hamiltonian of the system. In the following we will omit $H_{\rm sys}$ since we are interested only in the effect of decoherence on static quantum states. In writing Eq. (4) we have assumed that the system can decay in k channels, where the operator $\mathcal{J}_j$ is the ''jump operator'' corresponding to the jth decay channel. A jump operator can be any operator acting on the system. In the following we will focus on spontaneous emission. We will assume that the jump operator of a qubit undergoing spontaneous emission is $\propto|0\rangle\langle 1|$, where $|1\rangle$ is an excited state decaying to the ground state $|0\rangle$. The first term in Eq. (4) gives rise to a coherent, nonunitarian evolution under which pure states remain pure. The second term is a ''recycling term,'' which maps pure states into statistical mixtures.

B. Quantum trajectories

The master equation (4) corresponds to a physical situation where no measurements are performed on the environment to gain additional information about the system dynamics. However, in our context it is most useful to analyze the solution of (4) as an ensemble average over ''quantum trajectories'' of pure states in the sense of a stochastic Schrödinger equation [12]. Single trajectories may be viewed (with some caution) as the ''real'' quantum dynamics in a single experiment conditional to a ''measurement protocol'' of quantum jumps (t_1,j_1), (t_2,j_2), $\ldots$, (t_k,j_k), where a quantum jump of type j_i takes place at time t_i. The time evolution conditional to a given measurement protocol is called *a posteriori dynamics* [13]. Between quantum jumps the system evolves according to the effective Hamiltonian $H_{\rm eff}$:

$$U_{\rm eff}(t,t_0)=e^{-iH_{\rm eff}(t-t_0)}. \quad (5)$$

A quantum jump of type j leads to a wave-function collapse according to

$$|\Psi(t+dt)\rangle=\frac{\mathcal{J}_j|\Psi(t)\rangle}{\|\mathcal{J}_j|\Psi(t)\rangle\|}.$$

Thus the (unnormalized) state of the system conditional to a measurement protocol (t_1,j_1), (t_2,j_2), $\ldots$, (t_k,j_k), $t_{i+1}>t_i$, is

$$|\Psi(t)\rangle=U_{\rm eff}(t,t_k)\mathcal{J}_{j_k}U_{\rm eff}(t_k,t_{k-1})\mathcal{J}_{j_{k-1}}\cdots$$

$$\mathcal{J}_{j_1}U_{\rm eff}(t_1,t_0)|\Psi(t_0)\rangle. \quad (6)$$

The solution to the master equation (4) is obtained as an ensemble average over projectors onto states conforming to Eq. (6). The probability density for a certain sequence of quantum jumps to be realized is given by the square of the norm of the (unnormalized) state $|\Psi(t)\rangle$ according to Eq. (6). This is the basis of the quantum Monte Carlo wave function method [12].

C. Effects of decay and decoherence

According to the quantum trajectory formalism the (unperturbed) dynamics of the system is modified in the presence of a coupling to the environment by two different effects:

(a) *Quantum jumps*. For a given (normalized) state $|\Psi(t)\rangle$ a quantum jump of type $\mathcal{J}_j$ takes place in the time

interval $[t,t+dt)$ with probability $\|\mathcal{J}_j|\Psi(t)\rangle\|^2 dt$.

(b) *Effective time evolution*. Even if no quantum jump takes place, an (undesired) time evolution according to $U_{\text{eff}}(t,t_0)$ distorts the state of the system: the effective time evolution conditional to no quantum jump for an initial quantum state $|\Psi(t_0)\rangle$ is given by

$$|\Psi(t)\rangle=\frac{U_{\text{eff}}(t,t_0)|\Psi(t_0)\rangle}{||U_{\text{eff}}(t,t_0)|\Psi(t_0)\rangle||}. \tag{7}$$

As an example we consider the nine-qubit code given in Eq. (2) and assume an initial state $|\Psi(t_0)\rangle=\alpha|\underline{0}\rangle+\beta|\underline{1}\rangle$. Moreover, we assume that the ith qubit decays according to the jump operator $\mathcal{J}_i=\sqrt{\gamma}|0_i\rangle\langle 1_i|$, where γ is a decay rate. This is the decay model conforming to spontaneous emission. The non-Hermitian, effective Hamiltonian is thus given by

$$H_{\text{eff}}=-i\frac{\gamma}{2}\sum_{j=1}^{9}|1_j\rangle\langle 1_j|. \tag{8}$$

Thus the effective time evolution for times $\gamma t\gg 1$ reads

$$|\Psi(0)\rangle=\alpha|\underline{0}\rangle+\beta|\underline{1}\rangle$$

$$\xrightarrow{\gamma t\to\infty}|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle.$$

This example demonstrates that even if no quantum jump takes place, the encoded quantum superposition is changed in an undesired and unrecoverable way.

III. ERROR CORRECTION

Let $\mathcal{H}=\otimes^n\mathcal{H}_2$ be the Hilbert space of the system of n qubits and $\mathcal{H}_{\text{code}}\in\mathcal{H}$ be the subspace that can be stabilized against decoherence. The total Hilbert space $\mathcal{H}$ is decomposed into subspaces $\mathcal{H}_0=\mathcal{H}_{\text{code}}$ and unitarian images $\mathcal{H}_i=\mathcal{U}_i\mathcal{H}_{\text{code}}$ such that the subspaces are mutually orthogonal, i. e.,

$$\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_n \quad\text{and}\quad \mathcal{H}_i\perp\mathcal{H}_j(i\neq j).$$

Error correction takes place in two steps. First we perform a measurement that projects onto one of the subspaces $\mathcal{H}_i$. Secondly, in case of an error an appropriate unitary transformation depending on the outcome of the first step is performed to restore the initial state in $\mathcal{H}_{\text{code}}$. Thus, the subspaces $\mathcal{H}_i$ have to be designed such that for any state $|\Psi\rangle\in\mathcal{H}_{\text{code}}$ and any matrix A acting only on a single qubit

$$\mathcal{U}_i^{-1}\mathcal{P}_iA|\Psi\rangle\propto|\Psi\rangle \quad\text{or}\quad \mathcal{P}_iA|\Psi\rangle=0, \tag{9}$$

where $\mathcal{P}_i$ denotes the projector onto $\mathcal{H}_i$.

In the following we shall treat the two different sources of errors due to the effective time evolution and due to quantum jumps separately and show that both errors are corrected simultaneously. We shall assume in the following that we want to stabilize a quantum state $|\Psi\rangle\in\mathcal{H}_{\text{code}}$ for a time T and that the error correction procedure is applied N times.

A. Errors due to the effective time evolution

(a) *Example*. We first analyze the effect of error correction of a state that has evolved according to the effective Hamiltonian within the encoding given in Eq. (2). For the same decay conditions as in the example of Sec. II.C. the effective time evolution of the two encoded basis states is

$$U_{\text{eff}}(t)\begin{bmatrix}|\underline{0}\rangle\\|\underline{1}\rangle\end{bmatrix}=a_+^3(t)\begin{bmatrix}|\underline{0}\rangle\\|\underline{1}\rangle\end{bmatrix}+a_-^3(t)\begin{bmatrix}|\underline{1}\rangle\\|\underline{0}\rangle\end{bmatrix},$$

$$+a_+^2(t)a_-(t)\begin{bmatrix}|++-\rangle+|+-+\rangle+|-++\rangle\\|--+\rangle+|-+-\rangle+|+--\rangle\end{bmatrix},$$

$$+a_+(t)a_-^2(t)\begin{bmatrix}|--+\rangle+|-+-\rangle+|+--\rangle\\|++-\rangle+|+-+\rangle+|-++\rangle\end{bmatrix}. \tag{10}$$

Here we have defined $a_\pm(t)=(1\pm e^{-3\gamma t})/2$ and we have used the shortcut notation $|+++\rangle\equiv|+\rangle|+\rangle|+\rangle$, etc.

The error correction procedure applied to a quantum state that has undergone effective time evolution $U_{\text{eff}}(t)$ for some time t according to Eq. (10) changes the quantum state as follows: First the state is projected onto one of the following four subspaces by an appropriate measurement:

$$\mathcal{H}_{\text{code}}=\{|+++\rangle,|---\rangle\}=\{|\underline{0}\rangle,|\underline{1}\rangle\},$$

$$\mathcal{H}_1=\{|-++\rangle,|+--\rangle\},$$

$$\mathcal{H}_2=\{|+-+\rangle,|-+-\rangle\},$$

$$\mathcal{H}_3=\{|++-\rangle,|--+\rangle\}.$$

If the state is found in $\mathcal{H}_{\text{code}}$ no error is detected and the basis states after error correction read

$$\mathcal{P}_{\text{code}}U_{\text{eff}}(t)\begin{bmatrix}|\underline{0}\rangle\\|\underline{1}\rangle\end{bmatrix}=a_+^3(t)\begin{bmatrix}|\underline{0}\rangle\\|\underline{1}\rangle\end{bmatrix}+a_-^3(t)\begin{bmatrix}|\underline{1}\rangle\\|\underline{0}\rangle\end{bmatrix}. \tag{11}$$

By $\mathcal{P}_{\text{code}}$ we denote the projector onto $\mathcal{H}_{\text{code}}$. On the other hand, if the state is found in one of the Hilbert spaces $\mathcal{H}_i$ an appropriate unitary transformation $\mathcal{U}_i$ is applied to reconstruct the original state in $\mathcal{H}_{\text{code}}$. In this example, the basis states in the second (third) line of Eq. (10) are transformed into the correct (wrong) initial basis states ($i=1,2,3$):

$$\mathcal{U}_i\mathcal{P}_iU_{\text{eff}}(t)\begin{bmatrix}|\underline{0}\rangle\\|\underline{1}\rangle\end{bmatrix}=a_+^2(t)a_-(t)\begin{bmatrix}|\underline{0}\rangle\\|\underline{1}\rangle\end{bmatrix}+a_-^2(t)a_+(t)\begin{bmatrix}|\underline{1}\rangle\\|\underline{0}\rangle\end{bmatrix}. \tag{12}$$

Here $\mathcal{P}_i$ denotes the projector onto $\mathcal{H}_i$. If the error correction procedure is performed frequently on the time scale of decay, i.e., $\gamma T/N\ll 1$, we can expand $a_\pm(t)$ and write the time evolution operator of the system in basis $\{|\underline{0}\rangle,|\underline{1}\rangle\}$ after error correction corresponding to the two cases in Eq. (11) and Eq. (12) as follows:

$$\mathcal{P}_{\rm code}U_{\rm eff}(t)=\begin{bmatrix}1-9\gamma t/2 & 0\\ 0 & 1-9\gamma t/2\end{bmatrix}+o(\gamma^2t^2),$$

$$\mathcal{U}_i\mathcal{P}_iU_{\rm eff}(t)=\begin{bmatrix}3\gamma t/2 & 0\\ 0 & 3\gamma t/2\end{bmatrix}+o(\gamma^2t^2). \qquad (13)$$

Note that in both cases the time evolution operator is to first order in t proportional to the unit matrix. In the language of quantum trajectories the time evolution operator describing the state at time T is a product of time evolution operators $\mathcal{P}_{\rm code}U_{\rm eff}(T/N)$ and $\mathcal{U}_i\mathcal{P}_iU_{\rm eff}(T/N)$ from Eq. (13). The operators are chosen according to the probability for projection onto $\mathcal{H}_{\rm code}$ or $\mathcal{H}_i$. It can be seen easily that the deviation of the total time evolution operator from an operator proportional to the identity is of order $o(N(T/N)^2)$. Thus the error can be made arbitrarily small by choosing a sufficiently large N.

(b) *The general situation*. Let us now justify that this inhibition of the effective time evolution is in fact a generic feature of error correction. The effective, non-Hermitian Hamiltonian reads for the case of independent environments for each qubit:

$$H_{\rm eff}=H_{\rm eff}^{(1)}+\cdots+H_{\rm eff}^{(n)}, \qquad (14)$$

where $H_{\rm eff}^{(i)}$ is an effective Hamiltonian acting only on the ith qubit. Let us assume that in the time interval $[0,T)$ no quantum jump takes place. The conditional time evolution of a quantum state $|\Psi(0)\rangle\in\mathcal{H}_{\rm code}$ for the case of no error correction is given by Eq. (7). The quantum state immediately before each error correction event has evolved according to the time evolution operator

$$U_{\rm eff}(T/N)=\mathbb{1}-i(T/N)(H_{\rm eff}^{(1)}+\cdots+H_{\rm eff}^{(n)})+o((T/N)^2).$$

Now we shall consider the case of N error corrections within the interval $[0,T)$. Note that errors due to $U_{\rm eff}$, which are of first order in T/N are single qubit errors [cf. Eq. (3)] and can thus be corrected perfectly [cf. Eq. (9)]. The total time evolution operator at time T is given by a product of N operators, which are proportional to the identity up to first order in T/N. Thus the state of the system at time T after N error corrections is given by

$$|\Psi(T)\rangle\propto[\mathbb{1}+o((T/N)^2)]^N|\Psi(0)\rangle.$$

Therefore the accumulated error at time T is of order $o(N(T/N)^2)$ and can be made negligible by choosing a sufficiently large N.

B. Errors due to quantum jumps

The assumption of independent environments for the quantum bits implies that the jump operators $\mathcal{J}_j$ in the master equation (4) act only on single qubits. Thus single quantum jumps can be corrected perfectly according to (9). The probability P_1 for a (recoverable) single jump between two error correction events is according to Sec. II B.

$$P_1=\sum_{j=1}^{k}\int_0^{T/N}||U_{\rm eff}(T/N,t)\mathcal{J}_jU_{\rm eff}(t,0)|\Psi(0)\rangle||^2dt$$
$$=o(T/N),$$

whereas the probability $P_{>1}$ for a (nonrecoverable) multiple jump is of at least second order in T/N:

$$P_{>1}=1-||U_{\rm eff}(T/N,0)|\Phi(0)\rangle||^2-P_1=o((T/N)^2).$$

Thus the probability for a nonrecoverable error in the interval $[0,T)$ is of order $o(N(T/N)^2)$.

C. Combined correction of both kinds of errors

In practice both kinds of errors discussed in Secs. III A and III B. take place at the same time and have thus to be corrected simultaneously. However, the probability for an unrecoverable error due to the combination of an (by itself recoverable) error due the effective evolution and a (recoverable) error due to a single quantum jump is already of second order in T/N. Therefore, this probability can be made arbitrarily small by choosing an appropriate N too.

IV. HIGHER ORDERS

In practice, N cannot be made arbitrarily large since the required measurements and unitary transformations will take a finite time. Therefore, the order in T/N for both kinds of errors should be as large as possible. Of course codes capable to correct multiple errors due to quantum jumps will require more qubits and more complicated networks for error detection and correction [6,8]. By using similar arguments as in Sec. III A. it can be shown that the probability for an unrecoverable error due to the effective time evolution is at least of the same order as the error probability for an unrecoverable error due to quantum jumps. However, the actual order of the error probability for errors due to $U_{\rm eff}$ may be higher and depends on the design of the code as shown in the following two examples.

(a) *Constant weight code*. All vectors in $\mathcal{H}_{\rm code}\in H$ are superpositions of product states with factors $|0\rangle$ and $|1\rangle$. For the decay conditions presented in the example of Sec. II C., the effective time evolution of any of the states in the superposition depends only on the number of states $|1\rangle$. Thus, if all vectors in $\mathcal{H}_{\rm code}$ can be written as a superposition of product states with a fixed number of $|1\rangle$ states, the effective time evolution operator restricted to the subspace $\mathcal{H}_{\rm code}$ is proportional to the identity. Therefore, projection onto $\mathcal{H}_{\rm code}$ restores the original state without any error.

(b) *Codes from ''classical'' codes*. The construction of quantum error correcting given in [8] is based on two ''classical'' codes $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\{0\}\subset\mathcal{C}_2\subset\mathcal{C}_1$ [14]. It can be shown that after a single error correction event and in the absence of quantum jumps the error due to $U_{\rm eff}$ is of the order $o(t^d)$, where d is the minimum weight of the dual code $\mathcal{C}_2^{\perp}$ of $\mathcal{C}_2$ [15].

V. CONCLUSIONS

In this paper we have shown that existing error correction schemes can cope with quantum-optical systems with Markovian decay. This extends the scope of these schemes significantly and is the basis of their applicability for quantum computer models that seem realistic with present or planned technology [16–18].

Of course, the experimental realization of these schemes remains a challenging task. However, if specific decay models are considered, experimentally feasible schemes can be found [19].

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