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Quantum Information Theory
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Quantum Error Correction

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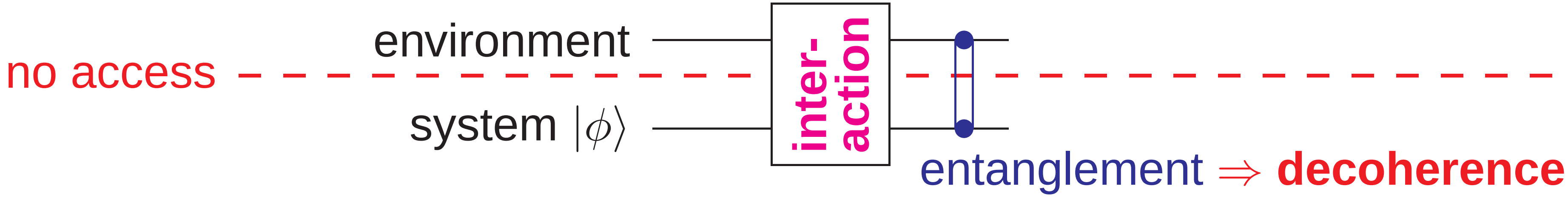
Outline

- general setting of quantum error correction
- approaches from
  - physics
  - discrete math/computer science
- bounds on QECC and optimal QECC
- non-qubit QECC
- qudits and finite fields
- encoding circuits
- graph codes
- conclusions

Quantum Error Correction

General scheme



Quantum Error Correction

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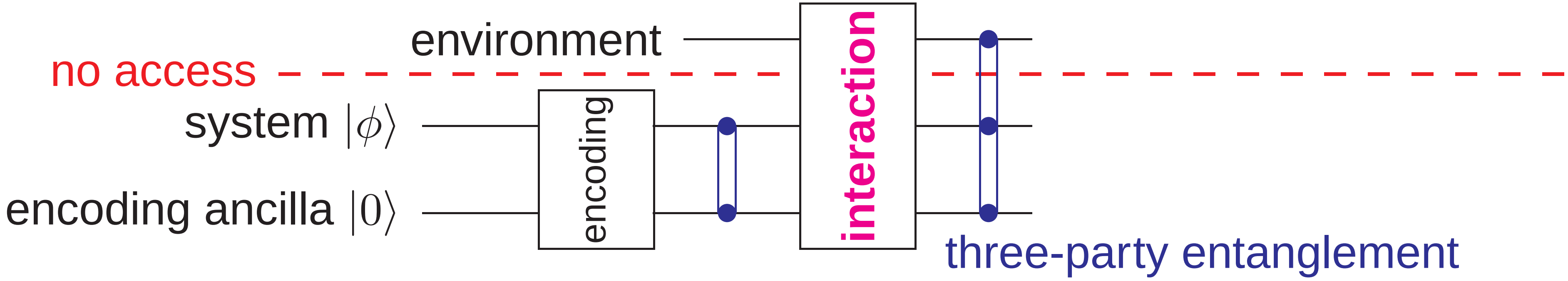
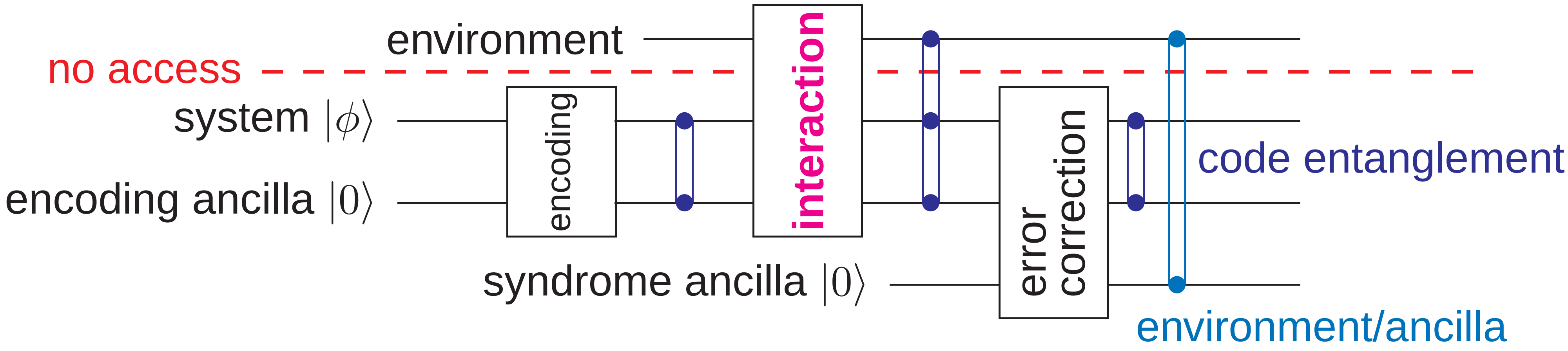


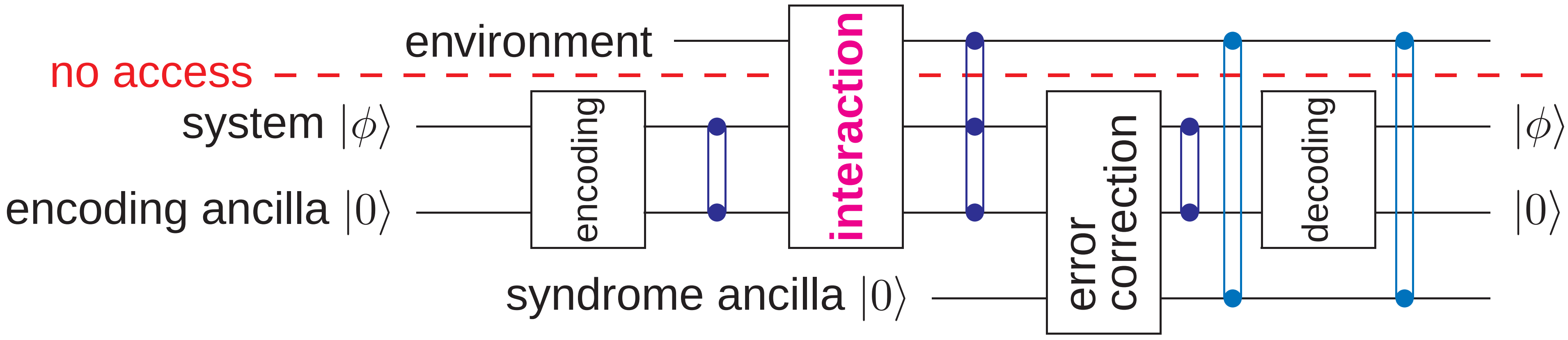
Quantum Error Correction

General scheme



Quantum Error Correction

General scheme



Quantum Error Correction

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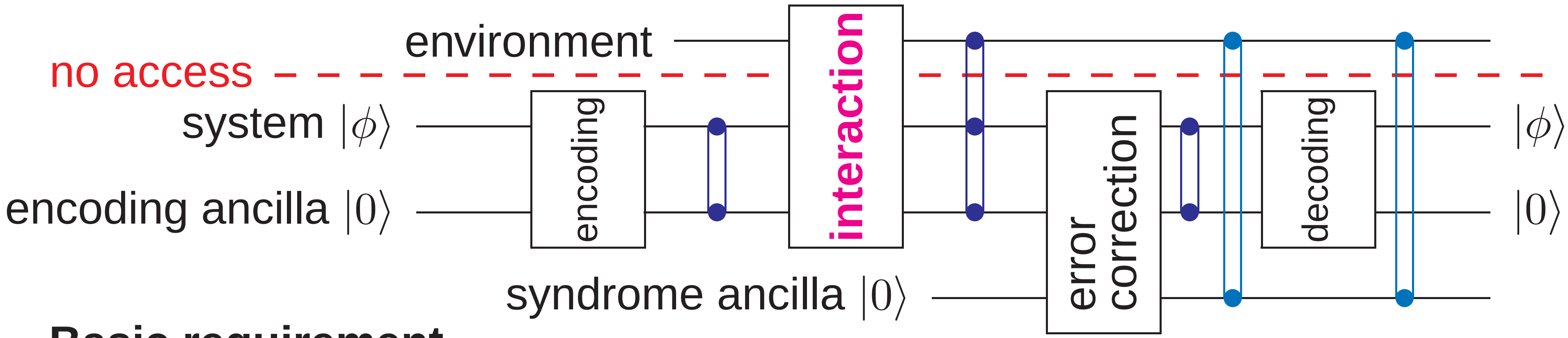


Basic requirement

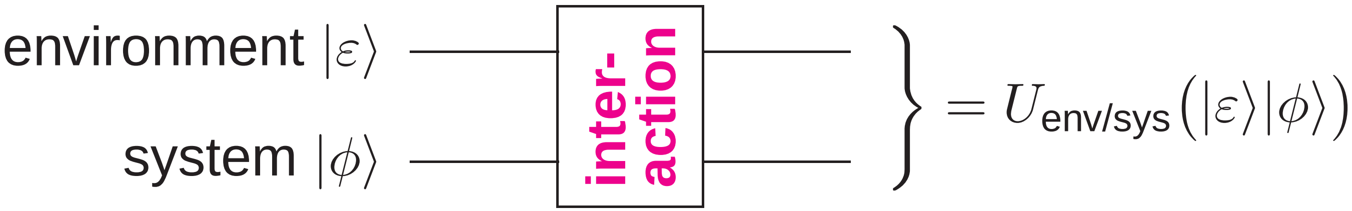
some knowledge about the interaction between system and environment

Common assumptions

- no initial entanglement between system and environment
- local or uncorrelated errors, i. e., only a few qubits are disturbed
  $\Longrightarrow$ CSS codes, stabilizer codes
- interaction with symmetry $\Longrightarrow$ decoherence free subspaces

Interaction System/Environment

“Closed” System



“Channel”

$$\mathsf{Q}\colon \rho_{\text{in}} := |\phi\rangle\langle\phi| \longmapsto \rho_{\text{out}} := \mathsf{Q}(|\phi\rangle\langle\phi|) := \sum_i A_i \rho_{\text{in}} A_i^\dagger$$

with Kraus operators (error operators) A_i

Local/low correlated errors

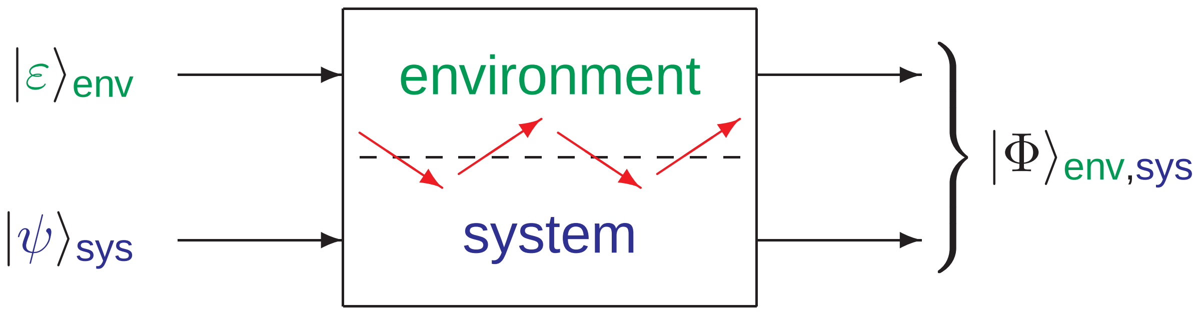
- product channel $\mathsf{Q}^{\otimes n}$ where Q is “close” to identity
- Q can be expressed (approximated) with error operators $\tilde{A}_i$ such that each A_i acts on few subsystems

Physics Approach: *Avoiding Errors*

see e. g. [Zanardi & Rasetti 97; Lidar; Knill] and many more

also called: noiseless subspaces/subsystems, passive error correction, error-avoiding codes

Main idea: *"Correct errors before they occur"*



known interaction (Hamiltonian)

Avoiding Errors: Mathematical Model

decomposition of the interaction algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$

$$\mathcal{A} \cong \bigoplus_j \mathbb{1}_{n_j} \otimes M(d_j, \mathbb{C}) \qquad \mathcal{H} \cong \bigoplus_j \mathbb{C}^{n_j} \otimes \mathbb{C}^{d_j}$$

irreducible components of dimension d_j and multiplicity n_j

$\Longrightarrow$ for $d_j = 1$ exists an decoherence free subspace of dimension n_j
(for $d_j > 1$ decoherence free subsystem)

Problem: requires non-trivial symmetry of the interaction

Designed Hamiltonians

Main idea: *"perturb the system to make it more stable"*

- fast (local) control operations
  $\Longrightarrow$ average Hamiltonian with more symmetry
  (cf. techniques from NMR, Hamiltonian simulation etc.)

- additional stabilizing Hamiltonian
  e. g. to suppress damping due to continuous monitoring the system
  (cf. e. g. [Ahn, Wiseman & Milburn, PRA 67, 052310 (2003)])

- combine stabilizing & "operation" Hamiltonian

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- combine stabilizing & "operation" Hamiltonian

Drawback: (for a computer scientist)

approach is highly dependent on the particular physical system

Computer Science Approach: *Discretize*

QECC Characterization

[Knill & Laflamme, PRA **55**, 900–911 (1997)]

A subspace $\mathcal{C}$ of $\mathcal{H}$ with orthonormal basis $\{|c_1\rangle, \ldots, |c_K\rangle\}$ is an error-correcting code for the error operators $\mathcal{E} = \{E_1, \ldots, E_\mu\}$, if there exists constants $\alpha_{k,l} \in \mathbb{C}$ such that for all $|c_i\rangle$, $|c_j\rangle$ and for all $E_k, E_l \in \mathcal{E}$:

$$\langle c_i | E_k^\dagger E_l | c_j \rangle = \delta_{i,j} \alpha_{k,l}. \qquad (1)$$

It is sufficient that (1) holds for a vector space basis of $\mathcal{E}$.

$\Longrightarrow$ only a finite set of errors

Error Basis

Pauli Matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- vector space basis of all 2×2 matrices
- unitary matrices which generate a *finite* group

Nice Unitary Error Basis [Knill, Klappenecker & Rötteler]

- set $\mathcal{E} := \{U_i : i = 1, \ldots, d^2\}$ of d^2 unitary matrices $U_i \in \mathcal{U}(d)$
- multiplicative structure of the indices

$$U_i U_j = \omega(i,j) U_{i \star j} = \omega(i,j) U_k, \quad \text{where } k = i * j \text{ in some group}$$

Example: Weyl-Heisenberg Group

- Generators: $$H_d := \langle X, Z \rangle$$

  where $X := \sum_{j=0}^{d-1} |(j+1) \bmod d\rangle\langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|$

  ($\omega_d := \exp(2\pi i/d)$)

- Relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- Basis:

$$H_d \big/ \zeta(H_d) = \left\{X^a Z^b \colon a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

Local Error Model

Error Basis for many Qudits

$\mathcal{E}$ error basis for subsystem of dimension d with $I \in \mathcal{E}$

$\Longrightarrow \mathcal{E}^{\otimes n}$ error basis with elements $E_1 \otimes \ldots \otimes E_n$, $E_i \in \mathcal{E}$

Code Parameters

$\mathcal{C} = [\![n, k, d]\!]$

n: number of subsystems used in total

k: number of (logical) subsystems encoded

d: "minimum distance"

– correct all errors acting on at most $(d-1)/2$ subsystems

– detect all errors acting on less than d subsystems

– correct all errors acting on less than d subsystems at known positions

No-Cloning Bound

Assumption: $\mathcal{C} = [\![n, 1, n/2+1]\!]$ exists

encoded state: $\sum_i \alpha_i |\psi_i\rangle|\phi_i\rangle$

splitting: $\sum_i \alpha_i^2 |\psi_i\rangle\langle\psi_i|$ | $\sum_i \alpha_i^2 |\phi_i\rangle\langle\phi_i|$

$\Downarrow$

padding: $\left(\sum_i \alpha_i^2 |\psi_i\rangle\langle\psi_i|\right) \otimes \left(|0\rangle\langle 0|\right)^{\otimes n/2}$ | $\left(|0\rangle\langle 0|\right)^{\otimes n/2} \otimes \left(\sum_i \alpha_i^2 |\phi_i\rangle\langle\phi_i|\right)$

$\Downarrow$

correction: $\sum_i \alpha_i |\psi_i\rangle|\phi_i\rangle$ | $\sum_i \alpha_i |\psi_i\rangle|\phi_i\rangle$

two independent copies

$\Longrightarrow$ no-cloning bound: $d - 1 < n/2$

Quantum Singleton Bound

(E. Rains, Nonbinary Quantum Codes, quant-ph/9703048)

Let $\mathcal{C} = [\![n, k, d]\!]_q$ be a quantum error-correcting code. Then

$$2d \leq n - k + 2. \tag{2}$$

If equality holds in (2) then $\mathcal{C}$ is pure and $\mathcal{C}$ is a **quantum MDS code**.

- bound is valid for arbitrary $q := \dim \mathcal{H}$
- for QECCs over qubits ($q = 2$), the bound is almost never achieved
- QECCs $[\![5, 1, 3]\!]_d$ exist for all dimensions d [Chau], [Rains]
- QECCs $[\![6, 2, 3]\!]_p$ and $[\![7, 3, 3]\!]_p$ exist for all primes $p \geq 3$ [Feng]

Quantum MDS Codes (QMDS)

- QMDS codes exist *for sufficiently large primes* p [Werner/Schlingemann]

- QMDS codes over $q = p^m$ exist [Grassl, Rötteler & Beth]
  - for all $n \leq q+1$ and all possible parameters k and d
  - for $q+1 < n \leq q^2+1$ with $d \leq q$ for *some* n (including q^2-1, q^2, q^2+1)
  - generalized Reed-Muller codes [Sarvepalli & Klappenecker] codes of length $n = q, 2q, 3q, \ldots, q^2$ with parameters

$$\mathcal{C} = [\![(\nu+1)q, (\nu+1)q - 2\nu - 2, \nu + 2]\!]_q$$

for $0 \leq \nu \leq q-2$

Bounds on Qubit-QECCs

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	2	1	1	**1**											
4	2	2	**2**	1	**1**										
5	3	**3**	2	1	1	**1**									
6	**4**	3	2	2	**2**	1	**1**								
7	3	3	2	2	2	1	1	**1**							
8	4	3	3	3	2	2	**2**	1	**1**						
9	4	3	3	3	2	2	2	1	1	1					
10	4	4	4	3	3	2	2	2	**2**	1	**1**				
11	5	5	4	3	3	3	2	2	2	1	1	**1**			
12	6	5	4	4	4	3	3	2	2	2	**2**	1	**1**		
13	5	5	4	4	4	3–4	3	3	2	2	2	1	1	**1**	
14	6	5	5	4–5	4	4	4	3	3	2	2	2	**2**	1	**1**

Bounds on $q = 4$ -QECCs

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	2	1	1	**1**											
4	**3**	2	**2**	1	**1**										
5	3	**3**	2	**2**	1	**1**									
6	**4**	3	**3**	2	**2**	1	**1**								
7	3	3	**3**	**3**	2	**2**	1	**1**							
8	4	**4**	**4**	3	**3**	2	**2**	1	**1**						
9	4	**4**	**4**	**4**	**3**	**3**	2	**2**	1	1					
10	4	4	4	**4**	**4**	**3**	**3**	2	**2**	1	**1**				
11	5	5	4	**4**	**4**	**4**	**3**	**3**	2	**2**	1	**1**			
12	6	5	4	4	4	**4**	**4**	**3**	**3**	2	**2**	1	**1**		
13	5	5	4	4	4	3–4	3	3	**3**	**3**	2	**2**	1	**1**	
14	6	5	5	4–5	4	4	4	**4**	**4**	**3**	**3**	2	**2**	1	**1**

(only new lower bounds by direct or trivial constructions are shown)

Non-Binary Quantum Codes

General Theory:

- Knill [1996], Klappenecker & Rötteler [2000]: Clifford codes
- Rains [1997], Hamada [2002]: codes for prime dimension
- Ashikhmin & Knill [2000]: codes for prime power dimension (CSS construction, $GF(q)$-linear codes, additive codes)

Specific Constructions:

- Aharonov & Ben-Or [1996]: polynomial codes with $k = 1$
- Chau [1997]: QECCs $[\![9, 1, 3]\!]_d$ and $[\![5, 1, 3]\!]_d$ for arbitrary d
- Bierbrauer [1998]: $GF(q)$-linear codes
- Grassl, Rötteler & Beth [2003]:
  quantum MDS codes for prime powers $q = p^\ell$

Qudits and Finite Fields (I)

Qudits

- tensor product of quantum systems of dimension d, in particular $d = p^m$, p prime

- *single qudit*

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \qquad \text{where } \alpha_i \in \mathbb{C} \text{ and } \sum_{i=0}^{d-1} |\alpha_i|^2 = 1$$

labels i of the basis states from an arbitrary set $\mathcal{A}$ with d elements, e. g.

$\mathcal{A} = \{0, 1, \ldots, d-1\}$ or $\mathcal{A} = \mathbb{F}_{p^m}$ for $d = p^m$, p prime

Qudits and Finite Fields (II)

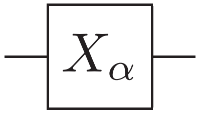
Finite Fields (Galois Fields)

- a field $\mathbb{F}_q$ with q elements exists if and only if $q = p^m$, p prime
- prime field $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$
- $\mathbb{F}_q$ is an m-dimensional vector space over $\mathbb{F}_p$
- $\mathbb{F}_q \cong \mathbb{F}_p[X]/(f(X)) = \{p(X) \bmod f(X) \colon p(X) \in \mathbb{F}_p[X]\}$
  where $f(X)$ is an irreducible polynomial of degree m
- trace: $\operatorname{tr} \colon \mathbb{F}_q \to \mathbb{F}_p$ where $\operatorname{tr}(\alpha) := \sum_{i=0}^{m-1} \alpha^{p^i} \in \mathbb{F}_p$
- in $\mathbb{F}_q$ exists a primitive $(q-1)$th root of unity

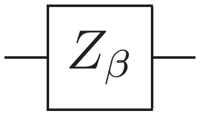
Elementary Gates (I)

One-Qudit Gates

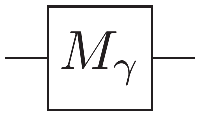
- $X_\alpha := \sum_{x \in \mathbb{F}_q} |x+\alpha\rangle\langle x|$ for $\alpha \in \mathbb{F}_q$



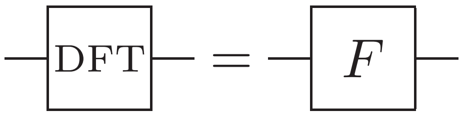
- $Z_\beta := \sum_{z \in \mathbb{F}_q} \omega^{\mathrm{tr}(\beta z)} |z\rangle\langle z|$ for $\beta \in \mathbb{F}_q$



- $M_\gamma := \sum_{y \in \mathbb{F}_q} |\gamma y\rangle\langle y|$ for $\gamma \in \mathbb{F}_q \setminus \{0\}$



- $\mathrm{DFT} := \frac{1}{\sqrt{q}} \sum_{x,z \in \mathbb{F}_q} \omega^{\mathrm{tr}(xz)} |z\rangle\langle x|$

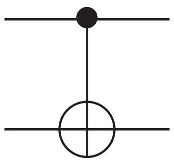


where $\omega := \exp(2\pi i/p) \in \mathbb{C}$

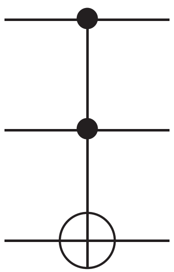
Elementary Gates (II)

Multi-Qudit Gates

- $\mathrm{ADD}^{(1,2)} := \sum_{x,y\in\mathbb{F}_q} |x\rangle_1|x+y\rangle_2\langle y|_2\langle x|_1$



- $\mathrm{HORNER}^{(1,2,3)} := \sum_{a,x,b\in\mathbb{F}_q} |a\rangle_1|x\rangle_2|ax+b\rangle_3\langle b|_3\langle x|_2\langle a|_1$



$\Longrightarrow$ universal for *classical* functions over $\mathbb{F}_q$

(Non-binary) Quantum Codes (QECC)

Error basis for *qudits*

[Ashikhmin & Knill, IEEE-IT **47**, 3065–3072 (2001)]

$$\mathcal{E} = \{X_\alpha Z_\beta \colon \alpha, \beta \in \mathbb{F}_q\}.$$

commutator relations:

$$X_\alpha Z_\beta = \omega^{-\operatorname{tr}(\alpha\beta)} Z_\beta X_\alpha$$

and

$$(X_\alpha Z_\beta)(X_{\alpha'} Z_{\beta'}) = \omega^{\operatorname{tr}(\alpha'\beta - \alpha\beta')}(X_{\alpha'} Z_{\beta'})(X_\alpha Z_\beta)$$

Non-binary Stabilizer Codes

similar constructions as for qubit-codes:

- error-group $G := \langle X_{\alpha_1} Z_{\beta_1} \otimes \ldots \otimes X_{\alpha_n} Z_{\beta_n} : \alpha_i, \beta_i \in \mathbb{F}_q \rangle$
- index group $G/\zeta(G) \cong (\mathbb{F}_q \times \mathbb{F}_q)^n$
- use classical error-correcting codes over finite fields
- additional constraint: self-orthogonal codes w. r. t. the symplectic inner product

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) * (\boldsymbol{\alpha}', \boldsymbol{\beta}') := \sum_{i=1}^{n} \operatorname{tr}(\alpha_i' \beta_i - \alpha_i \beta_i')$$

- abelian subgroups $S \leq G \Longleftrightarrow$ self-orthogonal codes

Example: Ternary QECC $\mathcal{C} = [\![9,5,3]\!]_3$

best QECC for qubits: $[\![9,5,2]\!]_2$, $[\![9,3,3]\!]_2$, $[\![11,5,3]\!]_2$

stabilizer matrix $(X|Z)$ of $\mathcal{C}$:

$$(X|Z) = \left(\begin{array}{ccccccccc|ccccccccc} 1 & 0 & 0 & 2 & 1 & 2 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 & 2 & 2 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 2 & 1 & 1 & 0 & 2 & 0 & 2 \end{array}\right)$$

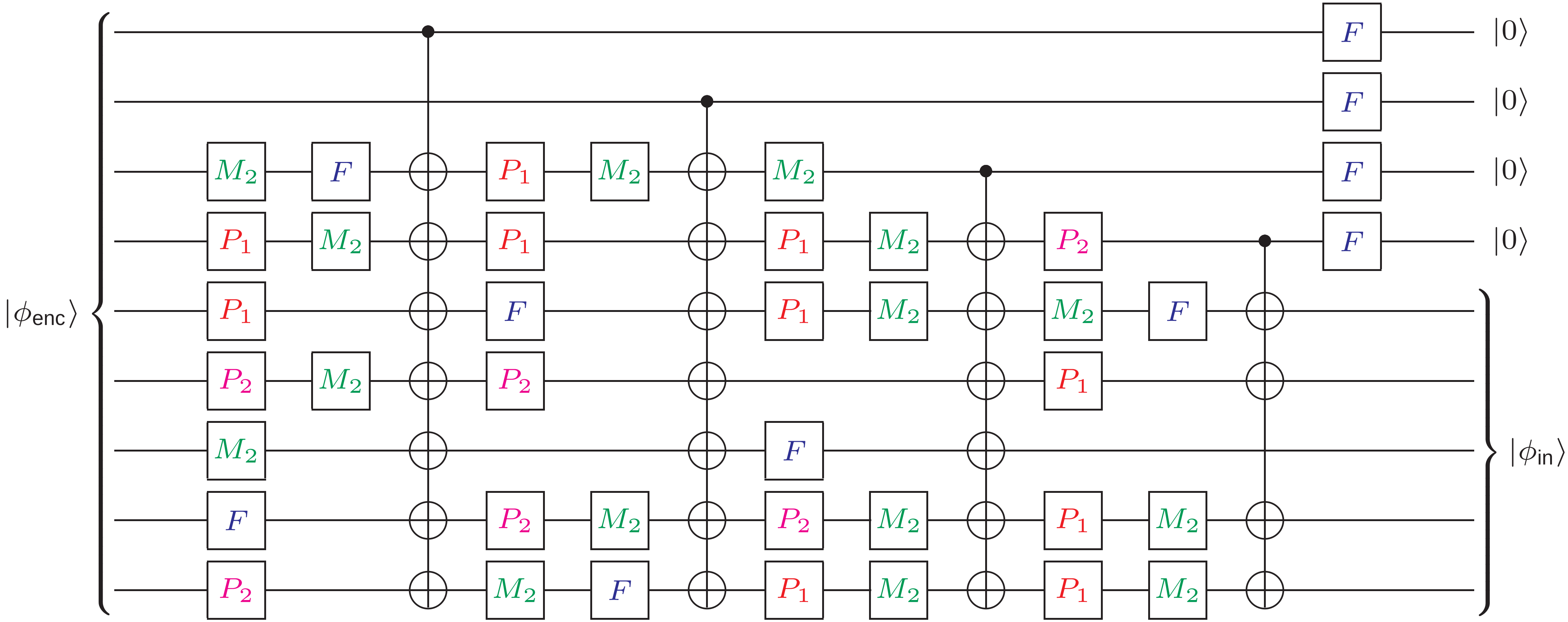
Encoding Stabilizer Codes

[Grassl, Rötteler & Beth, IJFCS, **14** (2003), pp. 757-775]

Basic idea: Use operations of the *generalized Clifford group* (or Jacobi group) to transform the stabilizer S into a trivial stabilizer $S_0 := \left\langle Z^{(1)}, \ldots, Z^{(n-k)} \right\rangle$.

- row/column operations on the matrix $(X|Z)$ to obtain "normal form" $(0|I0)$
- operations on $(X|Z)$ correspond to
  - "elementary" single-qudit gates
  - generalized CNOT-gate ADD
  - single qudit gate $P_\gamma := \sum_{y \in \mathbb{F}_q} \omega^{-\operatorname{tr}(\frac{1}{2}\gamma y^2)} |y\rangle\langle y|$ (for q odd; slightly different for q even)

(Inverse) Encoding Circuit for $\mathcal{C} = [\![9,5,3]\!]_3$



Equivalence of Graph Codes & Stabilizer Codes

Graph Codes:

[Schlingemann & Werner quant-ph/001211]

- k input nodes and n output nodes
- encoding via Hamiltonian:
  - edges in the graph correspond to coupling of subsystems
  - edge labels correspond to (integral) coupling constants
- measurement of the input nodes (cf. Teleportation)

Equivalence

1. graph codes/states are stabilizer codes/state [quant-ph/001211]
2. stabilizer codes are graph codes [Grassl, Klappenecker & Rötteler, Oct. 2001], [Schlingemann, quant-ph/0111080]

Algorithm

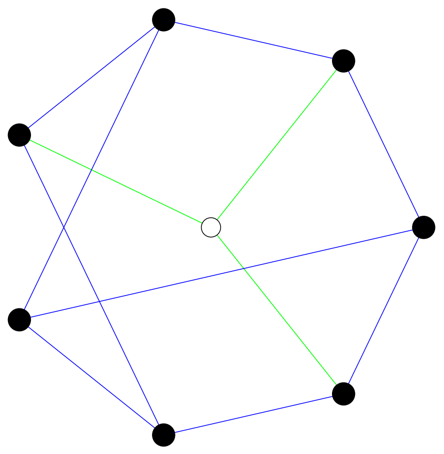
INPUT:	stabilizer matrix $G = (X\|Z) \in \mathbb{F}_q^{(n-k)\times 2n}$ of of $\mathcal{C} = [\![n,k,d]\!]_q$
OUTPUT:	graph corresponding to $\mathcal{C}$

- C self-orthogonal code generated by G with $C \subseteq C^*$
- find self-dual code D mit $C \subseteq D = D^* \subseteq C^*$
- generator matrix $G' := (X'|Z') = \left(\begin{array}{c|c} X & Z \\ \hline \widetilde{X} & \widetilde{Z} \end{array}\right)$ of D
- operations on rows & columns (excluding ADD) yield $G'' = (I|A)$
- D self dual $\Longrightarrow$ A symmetric $\Longrightarrow$ adjacency matrix of output vertices
- basis of $C^* \setminus D \Longrightarrow$ edges between input/output vertices

Example: CSS-Code $\mathcal{C} = [\![7,1,3]\!]_2$

$$G' = \left(\begin{array}{ccccccc|ccccccc} 1&0&0&1&0&1&1&&&&&&& \\ 0&1&0&1&1&1&0&&&&&&& \\ 0&0&1&0&1&1&1&&&&&&& \\ &&&&&&&1&0&0&1&0&1&1 \\ &&&&&&&0&1&0&1&1&1&0 \\ &&&&&&&0&0&1&0&1&1&1 \\ \hline 0&0&0&0&0&0&0&0&1&0&0&0&1&1 \end{array}\right) \cong \left(\begin{array}{ccccccc|ccccccc} 1&&&&&&&0&0&1&1&1&0&0 \\ &1&&&&&&0&0&1&0&1&0&0 \\ &&1&&&&&1&1&0&0&0&1&0 \\ &&&1&&&&1&0&0&0&0&1&1 \\ &&&&1&&&1&1&0&0&0&0&1 \\ &&&&&1&&0&0&1&1&0&0&0 \\ &&&&&&1&0&0&0&1&1&0&0 \end{array}\right)$$

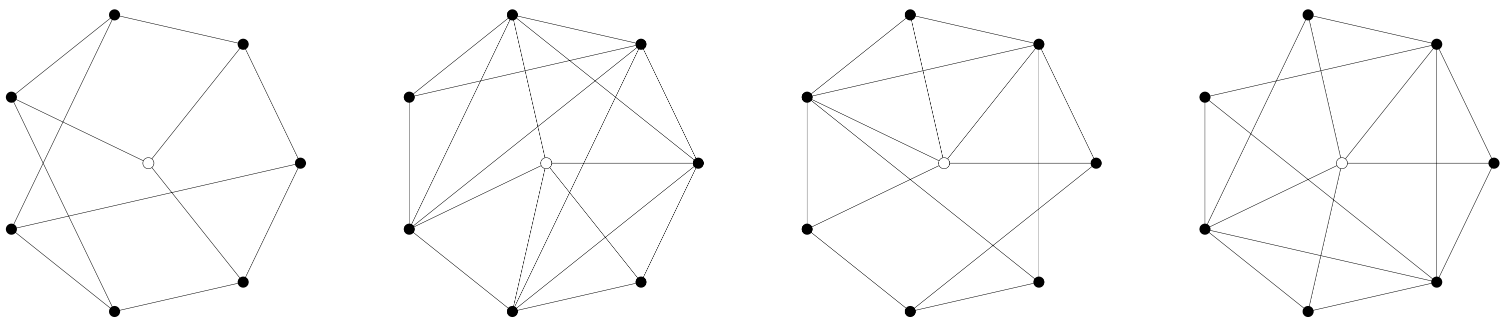
$$\Gamma_{\text{Hamming}} = \left(\begin{array}{c|ccccccc} 0&0&1&0&0&0&1&1 \\ \hline 0&0&0&1&1&1&0&0 \\ 1&0&0&1&0&1&0&0 \\ 0&1&1&0&0&0&1&0 \\ 0&1&0&0&0&0&1&1 \\ 0&1&1&0&0&0&0&1 \\ 1&0&0&1&1&0&0&0 \\ 1&0&0&0&1&1&0&0 \end{array}\right) \;\hat{=}$$



Graph is Not Unique

- transformation $G' \to (I|A)$ not unique
- basis of $D \setminus C$ not unique

$\Longrightarrow$ different (inequivalent) graphs for equivalent codes



$\Longrightarrow$ no obvious correspondence of graph properties and code properties

$\Longrightarrow$ alternative realization of a *stabilizer code* as graph code

Advantages of Non-Qubit QECC

- a single (abstract) particle is hard to control, but
  it has more than two degrees of freedom
  $\Longrightarrow$ use all of them

- non-qubit QECC have a higher minimum distance
  $\Longrightarrow$ they can tolerate errors on more subsystems

- detection of leakage into "unused" degree of freedom yields information about error positions
  $\Longrightarrow \mathcal{C} = [\![n, k, d]\!]_q$ can correct errors at $d-1$ *known* positions

- final example: 20 qubits:

qubit code	$\mathcal{C}_1 = [\![20, 12, 3]\!]_2$	correction of a single error
quaternary code	$\mathcal{C}_2 = [\![10, 6, 3]\!]_4$	errors at positions $2i-1$ and $2i$

  $\Longrightarrow$ even for qubits, non-qubit codes can be better

Outlook

- optimizing codes for specific channels
- good codes for fault tolerant operations
- combining passive error stabilization and active error correction

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