Probability Density Function of the Line–of–Sight Angle Error

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Abstract — **The closed form probability density function of the angle between the input and output of the time discrete AWGN channel is derived.**

I. Introduction

Consider two N–dimensional carthesian systems $\mathcal{N}$ and $\mathcal{Y}$. In the origin $\mathbf{0}_1$ of the first one is the true (most probable) position of some remote object O observed from the origin $\mathbf{0}_2$ of the second one. The object randomly fluctuates around $\mathbf{0}_1$ so that in each coordinate $\vec{\mathcal{N}}_i$; $i = 1, \ldots, N$, of $\mathcal{N}$ it randomly and independently changes its position according to identical Gaussian probability density functions with zero mean and variance σ^2. Without loss of generality it can be assumed that $\overline{\mathbf{0}_1\mathbf{0}}_2 = 1$ and that $\vec{\mathcal{N}}_1$ is collinear with $\vec{\mathcal{Y}}_1$. *The Line–of–Sight Angle Error* (LOSE) *to a remote object* [1] is the random angle ϕ between the line–of–sight $\overline{\mathbf{0}_2\mathrm{O}} = \boldsymbol{y}$ and the unit radius–vector of the true position $\overrightarrow{\mathbf{0}_1\mathbf{0}}_2 = \boldsymbol{x}$.

Determination of the probability density function (pdf) $p_{\boldsymbol{y}}(\phi|\boldsymbol{x})$ of the LOSE can be considered as information theoretic problem to determine the angle between the input $\boldsymbol{x}$ and output $\boldsymbol{y}$ of the additive white Gaussian noise (AWGN) channel with discrete time, where the signal–to–noise ratio is given by $SNR = \frac{1}{N\sigma^2}$. For $N = 1$ obviously $\phi \equiv 0$. For $N = 2$ the pdf $p_{\boldsymbol{y}}(\phi|\boldsymbol{x})$ is well known.

In [1] F. S. Weinstein derived the explicit form of $p_{\boldsymbol{y}}(\phi|\boldsymbol{x})$ for $N = 3$. To our knowledge until now there exists no explicit solution of $p_{\boldsymbol{y}}(\phi|\boldsymbol{x})$ (expressed by elementary functions and the error function and without any approximations) for $N > 3$.

In this paper we present the general solution in the following form

$$
\begin{aligned}
p_{\boldsymbol{y}}(\phi|\boldsymbol{x}) = {} & \left(\frac{N \cdot SNR}{2\exp(SNR)}\right)^{\frac{N}{2}} \frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{N-1}{2}\right)} \sin^{N-2}(\phi)\, \frac{(-1)^{N-1}}{2} \sqrt{\frac{\pi}{\frac{N\cdot SNR}{2}}} \\
& \cdot \left[\frac{(N-1)!}{2^{N-1}} \exp\left(\frac{(N\cdot SNR\cos\phi)^2}{4\frac{N\cdot SNR}{2}}\right) \operatorname{erfc}\left(\frac{(N\cdot SNR\cos\phi)}{2\sqrt{\frac{N\cdot SNR}{2}}}\right) \right. \\
& \cdot \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \frac{\frac{N\cdot SNR}{2}^{k-(N-1)} (N\cdot SNR\cos\phi)^{N-1-2k}}{k!(N-1-2k)!} \\
& - \frac{(N-1)!}{2^{N-2}\sqrt{\frac{N\cdot SNR}{2}}\pi} \sum_{\nu=0}^{N-2} \left(\sum_{k=0}^{\lfloor \nu/2 \rfloor} \frac{\cos^{\nu-2k}\phi}{\left(\frac{N\cdot SNR}{2^\nu}\right)^k k!(\nu-2k)!} \right. \\
& \left.\left. \cdot \sum_{k=0}^{\lfloor \frac{N-2-\nu}{2} \rfloor} \frac{\left(-\frac{2}{N\cdot SNR}\right)^k (-2\cos\phi)^{N-2-\nu-2k}}{(N-1-\nu)k!(N-2-\nu-2k)!} \right) \right]. \qquad (1)
\end{aligned}
$$

II. Derivation of the pdf

The probability density function of the channel output vector $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n}$, on condition that $\boldsymbol{x}$ was transmitted, is

$$
p_{\boldsymbol{y}}(y_1, \ldots, y_N|\boldsymbol{x}) = \left(\frac{N\cdot SNR}{2\pi}\right)^{\frac{N}{2}} \exp\left(-\frac{1 - 2|\boldsymbol{y}|\cos\phi + |\boldsymbol{y}|^2}{2\,(N\cdot SNR)^{-1}}\right),
$$

where ϕ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Transformation of the coordinate system $\mathcal{Y}$ to the N–dimensional spherical coordinate system $\mathcal{S} = (\mathbb{R}, \Phi_1, \ldots, \Phi_{N-1})$ and integration of all variables except of $\phi \in \Phi_1$ leads to

$$
\begin{aligned}
p_{\boldsymbol{y}}(\phi|\boldsymbol{x}) = {} & \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{N\cdot SNR}{2\pi}\right)^{\frac{N}{2}} \sin^{N-2}(\phi) \exp\left(-\frac{N\cdot SNR}{2}\right) \\
& \cdot \int_0^\infty \rho^{N-1} \exp\left(\frac{2\rho\cos\phi - \rho^2}{2} N\cdot SNR\right) d\rho. \qquad (2)
\end{aligned}
$$

In order to obtain equation (1) the integral in equation (2) has to be solved. In [3] the following formula is presented

$$
\begin{aligned}
& \int_0^\infty x^n \exp\left(-px^2 - qx\right) dx = \\
& \qquad \frac{(-1)^n}{2} \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \left[\exp\left(\frac{q^2}{4p}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right) \right]. \qquad (3)
\end{aligned}
$$

The partial derivatives in equation (3) can be expressed by a recursive formula which leads to the nonlinear difference equation

$$
y_{n+2} - a y_{n+1} - b(n+1) y_n = 0, \qquad (4)
$$

where a and b are independent of n. We solved this nonlinear difference equation using the method of exponential generating functions. By replacing this solution in (2) equation (1) is obtained.

III. Conclusion

The obtained pdf in the closed form (1) can be used for calculation of the corresponding cumulative distribution function (cdf) which determines the best known general lower and upper bounds on the decoding error probability of optimal block codes on the AWGN channel [2]. For odd values of $N \geq 3$ this cdf reduces to a closed form too.

References

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[3] Prudnikov, A. P., J. A. Bickov, O. I. Maricev, *Integrals and Series (Russian)*, Nauka, Moscow, 1981.